Logic proofs in FOPL and FOPLN

Supplementary material to the paper
Bunch Theory: logic, axioms, applications and model

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July 20, 2023

Abstract

This document provides supplementary material to our paper "Bunch Theory: logic, axioms, applications and model". In that paper we note that classical first order predicate logic (FOPL) is not suitable for use with our theory and we introduce FOPLN, a logic which can accomodate the concept of "nothing" as a mathematical entity. Here our aim is to show that FOPLN presents no or minimal additional proof burdens. To do that we take an extensive predicate logic toolkit and prove its laws in both FOPL and FOPLN, We will see we can essentially prove the same laws, with just two occasions in which FOPLN requires an additional definedness condition. We include a logic primer which explains our graphical method of presenting proofs as trees based on a combination of backward and forward reasoning.

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1 Introduction

Eric Hehner uses a radical reformulation of set theory, in which the collection and packaging of elements are seen as separate activities. This provides for unpackaged collections, referred to as "bunches". Of particular interest is the contents of the empty set, which we call *null* and which captures the idea of "nothing" as a mathematical entity. Expressions that would be considered as undefined in conventional mathematics denote *null* in our approach, that is they literally denopte "nothing". As an example $1/0 = null$. As we discuss in our paper, in classical first order predicate logic (FOPL) we can prove there exists an element equal to $1/0$, and this result is inconsistent with our theory, so we need a different logic. In FOPLN we have an additional predicate $\delta(E)$ which is true iff E is an element, and we have different quantifier rules from FOPL. The propositional and equality rules, on the other hand, are identical for both logics.

This document is organised as follows. In section 2 we present a logic primer in which we discuss sequent calculus, our graphical style of proof presentation, backward vs forward reasoning, and poof by contradicion. In section 3 we list the laws from our logic toolkit, in section 4 we give proofs of these laws in both FOPL and FOPLN. The full set of inference rules for FOPLN (including propositional logic rules) are given in an Appendix A, as are the inference rules of FOPL which differ from those of FOPLN.

2 Logic primer

2.1 Logic proofs: forward and backward reasoning

We illustrate our approach to proof presentation using propositional logic, which is identical for FOPL and FOPLN. We use the sequent calculus style of Natu-

ral Deduction, and introduce a graphical notation for both inference rules and proofs.

A sequent has the form $HYP \vdash P$ where HYP is a list of hypotheses we are currently assuming, and P is a proposition. The sequent asserts that there is a proof of P under the assumptions of HYP.

As an example of an inference rule from propositional logic we consider the $\Leftrightarrow$-intro rule. This tells us that if we can prove $P \Rightarrow Q$ under hypotheses HYP and we can also prove $Q \Rightarrow P$ under the same hypotheses, then we have a proof of $P \Leftrightarrow Q$ under these hypotheses. The rule is:

$$\frac{HYP \vdash P \Rightarrow Q, \quad HYP \vdash Q \Rightarrow P}{HYP \vdash P \Leftrightarrow Q} \quad \Leftrightarrow\text{-intro}$$

The two sequents on the top line of the rule are its premises, and the single sequent below the line is our conclusion.

Such a rule can be used in two ways. Reasoning in a forward direction, if we have established both $P \Rightarrow Q$ and $Q \Rightarrow P$ the rule allows us to conclude $P \Leftrightarrow Q$. Reasoning in a backward direction and with a requirement to establish the goal $P \Leftrightarrow Q$. the rule allows us to split our goal into two sub-goals: establishing $P \Rightarrow Q$ and establishing $Q \Rightarrow P$. An alternative presentation of the rule is tree form is as follows.

$$\begin{array}{c} P \Rightarrow Q \\ \Leftrightarrow\text{-intro} \\ \diagup \quad \diagdown \\ P \Rightarrow Q \qquad Q \Rightarrow P \end{array}$$

In this presentation of the rule, HYP seems to have disappeared. In fact the hypotheses we argue under will be visible when we present a full tree proof, are only mentioned at the nodes of the truth tree at a point where they change. This is the case with the deduction rule.

$$\frac{HYP, P \vdash Q}{HYP \vdash P \Rightarrow Q} \quad \text{DED}$$

Which in graphical form we show as:

$$\begin{array}{c} P \Rightarrow Q \\ \text{DED} \\ \{P\} \\ | \\ Q \end{array}$$

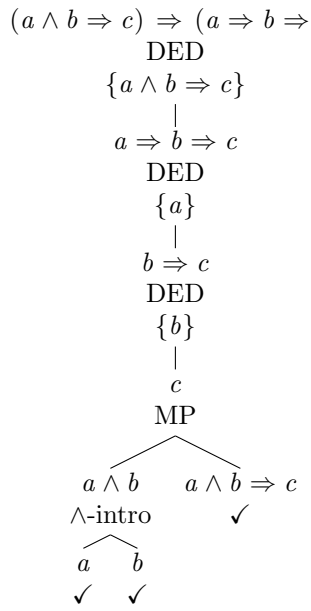
This rule converts a proof goal $P \Rightarrow Q$ into a sub-goal P which may be proved *under the assumption* of P.

Now, recalling that implication $\Rightarrow$ associates to the right and ha a higher precedence that equivalence $\Leftrightarrow$ consider the proof of:

$a \wedge b \Rightarrow c \Leftrightarrow a \Rightarrow (b \Rightarrow c)$

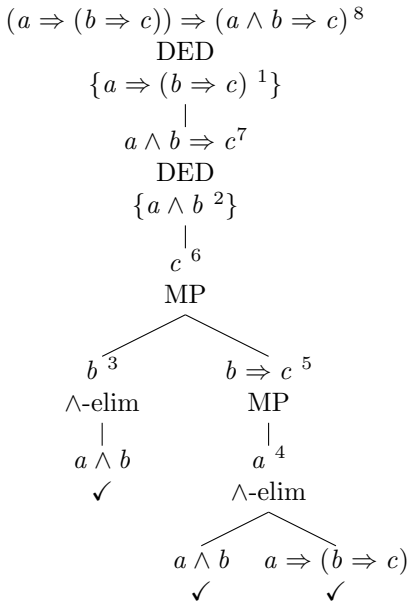
Here the first step would be to split the proof into LHS $\Rightarrow$ RHS and RHS $\Rightarrow$ LHS using the $\Leftrightarrow$-intro rule. We will consider this has been done, and show a

proof of LHS $\Rightarrow$ RHS. Before studying the proof the dedicated reader might like to look at the rules for MODUS PONENS (MP) and $\wedge$-intro as given in sequent calculus form in the appendix A, and convert these to inverted (graphical) form.



The top goal in this proof is $(a \wedge b \Rightarrow c) \Rightarrow (a \Rightarrow b \Rightarrow c)$. We use the deduction rule DED to convert this to proving the sub-goal $a \Rightarrow b \Rightarrow c$ under the assumption of $(a \wedge b \Rightarrow c)$. DED is applied two more times, generating new goals and adding to the assumptions under which we are reasoning. When we have generated c as a proof goal, we use the modus ponens rule. Here we are reasoning under the assumptions introduced by the above applications od the deduction rule DED. The MP rule is used to split the current goal c into the two sub goals b and $b \Rightarrow c$, and so on. In this proof we reach a leaf note when the goal is already present in our assumptions. These can just be ticked off. Other possible ways of arriving at a leaf node in a proof are when the goal is a result already proved, a goal of the form $E = E$ which can be discharged by the equality axiom, or more generally a goal that can be discharged by any axiom.

After studying this proof we invite the reader to try the proof of RHS $\Rightarrow$ LHS. We give our solution below, and we use it to illustrate our next topic, which is the conversion of backward tree proofs to forward linear proofs, which are a more classic form of proof presentation. The numbers which annotate each node are for use in this process.



To convert the above tree proof to a linear proof we perform a tree traversal, and we have chosen to do so in a clockwise direction. As we move down the tree we number assumptions. As we come back up we number proof goals. The numbers 1 to 8 which annotate our graph correspond to lines 1 to 8 of a linear proof. In the proof, to show the tree structure, we indent after each assumption, and outdent at the line corresponding to the node at which the assumption is discharged. This yields the following linear proof.

1.	$a \Rightarrow (b \Rightarrow c)$	Assumption
2.	$\quad a \wedge b$	Assumption
3.	$\quad\quad b$	2, ∧-elim
4.	$\quad\quad a$	2, ∧-elim
5.	$\quad\quad b \Rightarrow c$	3,5,MP
6.	$\quad\quad c$	4,1,MP
7.	$\quad a \wedge b \Rightarrow c$	6,DED
8.	$(a \Rightarrow (b \Rightarrow c)) \Rightarrow (a \wedge b \Rightarrow c)$	7,DED

2.2 Combining backward and forward reasoning.

We can combine backward and forward reasoning by using forward reasoning at a node to add to the hypotheses under which we are reasoning. For example id application of the deduction rule adds $a \wedge b$ to our hypotheses, we may assume it also adds a and b as separate hypotheses. In our logic toolbox proofs we will limit ourselves to simple uses of forward reasoning, and we will not convert our tree proofs to linear form. However, for completeness we present here a more extensive example provided by Dr Campbell Ritchie.

A student wakes up and hears no traffic noise. They know that there is no traffic on Saturday, there is no traffic on Sunday, and that there are no classes on Saturday or Sunday. Show how they can reason that no traffic implies no

classes.

We formalise this as:

$(sa \Rightarrow nc) \wedge (su \Rightarrow nc\} \wedge (nt \Rightarrow sa \vee su) \Rightarrow (nt \Rightarrow nc)$

In the proof below we have numbered the assumptions generated at each node in the style of a linear proof.

$$(sa \Rightarrow nc) \wedge (su \Rightarrow nc) \wedge (nt \Rightarrow sa \vee su) \Rightarrow (nt \Rightarrow nc)\ ^{10}$$

DED

1	$(sa \Rightarrow nc) \wedge (su \Rightarrow nc) \wedge (nt \Rightarrow sa \vee su)$	*Assumption*
2	$nt \Rightarrow sa \vee su$	$1 \wedge -elim$
3	$(sa \Rightarrow nc) \wedge (su \Rightarrow nc)$	$2 \wedge -elim$
4	$sa \Rightarrow nc$	$3 \wedge -elim$
5	$su \Rightarrow nc$	$3 \wedge -elim$

$$|$$

$$nt \Rightarrow nc\ ^{9}$$

DED

6	nt	*Assumption*
7	$sa \vee su$	$2, 6, MP$
8	nc	$4, 5, 7 \vee -elim$

$$|$$

$$nc$$

$$\checkmark$$

2.3 Proof by contradiction (reductio ad absurdam).

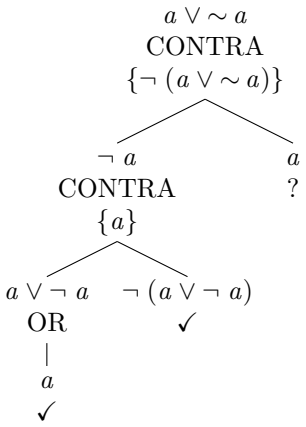
One advantage that FOPL and FOPLN share over multi-valued logics is the straightforward presentation of proof by contradiction. This technique discharges a proof goal by assuming its opposite is true and, with the help of this assumption, deriving a contradiction by discharging as sub-goals both Q and $\neg\ Q$ for some Q of our choice.

This technique is used frequently in our logic toolbox proofs. We illustrate it here by proving the "law of the excluded middle" $a \vee \neg\ a$.

First we look at the two inference rules for CONTRA. In our inverted tree form we can represent them as follows

$$\begin{array}{c} P \\ \text{CONTRA} \\ \{\neg\ P\} \\ \diagup\quad\diagdown \\ Q \quad \neg\ Q \end{array} \qquad\qquad \begin{array}{c} \neg\ P \\ \text{CONTRA} \\ \{P\} \\ \diagup\quad\diagdown \\ Q \quad \neg\ Q \end{array}$$

To prove the law of the excluded middle we apply the appropriate CONTRA rule, and obtain a new hypothesis $\neg\ (a \vee \neg\ a)$ under which we prove the opposing goals a and $\neg\ a$. We leave completion of this proof to the dedicated reader.



3 Predicate logic rules in inverted form.

Before proving, in inverted form, the laws from a predicate logic toolbox, we present the predicate logic rules of FOPLN and FOPL in inverted from and prove an important derived law of FOPLN.

3.1 $\forall$-intro

FOPL	FOPLN
$\forall\, x \bullet P$	$\forall\, x \bullet P$
$\forall$-intro	$\forall$-intro
|	$\delta(\alpha)$
$P[\alpha/x]$	|
	$P[\alpha/x]$

3.2 $\forall$-elim

FOPL	FOPLN
$P[E/x]$	$P[E/x]$
$\forall$-elim	$\forall$-elim
|	$\delta(E)$ $\quad$ $\forall\, x \bullet P$
$\forall\, x \bullet P$	

3.3 ∃-intro

FOPL

$$\begin{array}{c} \exists\, x \bullet P \\ \exists\text{-intro} \\ | \\ P[E/x] \end{array}$$

FOPLN

$$\begin{array}{c} \exists\, x \bullet P \\ \exists\text{-intro} \\ \diagup\ \diagdown \\ \delta(E) \quad P[E/x] \end{array}$$

3.4 ∃-elim

FOPL

$$\begin{array}{c} Q \\ \exists\text{-elim} \\ \diagup\ \diagdown \\ P[\alpha/x] \Rightarrow Q \quad \exists\, x \bullet P \end{array}$$

FOPLN

$$\begin{array}{c} Q \\ \exists\text{-elim} \\ \delta(\alpha) \\ \diagup\ \diagdown \\ P[\alpha/x] \Rightarrow Q \quad \exists\, x \bullet P \end{array}$$

3.5 The derived FOPLN law ∃-intro-mu

We can derive the following law for FOPLN, which we name after the auther who suggested it.

$$\frac{HYP,\, \delta(\alpha) \vdash P[\alpha/X]}{HYP \vdash \exists\ x \bullet P}\ \exists\text{-intro-mu}$$

Its inverted form and derivation are shown below. For the derivaton, we use backward reasoning. We first apply the ∃-elim rule to our goal. Our original goal then reappears as a sub-goal to which we apply the ∃-intro rule. The inverted form of the rule is drived by pruning out those branches for which we can discharge all leaf nodes. Below we see the resulting rule in inverted from on the left and its derivation on the right.

$$\begin{array}{c} \exists\, x \bullet P \\ \exists\text{-intro-mu} \\ \{\delta(\alpha)\} \\ | \\ P[\alpha/x] \end{array}$$

$$\begin{array}{c} \exists\, x \bullet P \\ \exists\text{-elim} \\ \{\delta(\alpha)\} \\ \diagup\ \diagdown \\ P[\alpha/x] \Rightarrow (\exists\, x \bullet P) \quad \exists\, x \bullet P \\ \text{DED} \qquad\qquad \exists\text{-intro} \\ \{P[\alpha/x]\} \qquad\qquad \diagup\ \diagdown \\ | \qquad\qquad \delta(\alpha) \quad P[\alpha/x] \\ \exists\, x \bullet P \qquad\qquad \checkmark \\ \exists\text{-intro} \\ \diagup\ \diagdown \\ \delta(\alpha) \quad P[\alpha/x] \\ \checkmark \quad \checkmark \end{array}$$

4 The laws from our logic toolbox

deMorgan
1. $(\neg\ \forall x \bullet P) \Leftrightarrow (\exists x \bullet \neg P)$
2. $(\forall x \bullet \neg P) \Leftrightarrow \neg (\exists x \bullet P)$

Change of order of bound variables, assuming P admits y for x. [1]
3. $(\forall x \bullet \forall y \bullet P) \Leftrightarrow (\forall y \bullet \forall x \bullet P)$
4. $(\exists x \bullet \exists y \bullet P) \Leftrightarrow (\exists y \bullet \exists x \bullet P)$

Splitting
5. $(\forall x \bullet P \wedge Q) \Leftrightarrow (\forall x \bullet P) \wedge (\forall x \bullet Q)$
6. $(\exists x \bullet P \vee Q) \Leftrightarrow (\exists x \bullet P) \vee (\exists x \bullet Q)$
7. $(\exists x \bullet P \wedge Q) \Leftrightarrow (\exists x \bullet P) \wedge (\exists x \bullet Q)$

Assuming x is not free in Q
8. $(\forall x \bullet Q) \Leftrightarrow Q$
9. $(\exists x \bullet Q) \Leftrightarrow Q$
10. $(\forall x \bullet P) \vee Q \Leftrightarrow (\forall x \bullet P \vee Q)$ 11. $(\exists x \bullet P) \wedge Q \Leftrightarrow (\exists x \bullet P \wedge Q)$ 12. $(\forall x \bullet Q \Rightarrow P) \Leftrightarrow Q \Rightarrow (\forall x \bullet P)$
13. $(\exists x \bullet P \Rightarrow Q) \Leftrightarrow (\forall x \bullet P) \Rightarrow Q$
14. $(\forall x \bullet P \Rightarrow Q) \Leftrightarrow (\exists x \bullet P) \Rightarrow Q$
15. $(\exists x \bullet Q \Rightarrow P) \Leftrightarrow Q \Rightarrow (\exists x \bullet P)$

Change of bound variable name
16. $(\forall x \bullet P) \Leftrightarrow (\forall y \bullet P[y/x])$
17. $(\exists x \bullet P) \Leftrightarrow (\exists y \bullet P[y/x])$

Monotonicity
18. $(\forall x \bullet P \Rightarrow Q) \Leftrightarrow ((\forall x \bullet P) \Rightarrow (\forall x \bullet Q))$
19. $(\forall x \bullet P \Rightarrow Q) \Leftrightarrow ((\exists x \bullet P) \Rightarrow (\exists x \bullet Q))$

Equivalence
20. $(\forall x \bullet P \Leftrightarrow Q) \Rightarrow ((\forall x \bullet P) \Leftrightarrow (\forall x \bullet Q))$
21. $(\forall x \bullet P \Leftrightarrow Q) \Rightarrow ((\exists x \bullet P) \Leftrightarrow (\exists x \bullet Q))$
22. $(\exists x \bullet P \Rightarrow Q) \Leftrightarrow ((\forall x \bullet P) \Rightarrow (\exists x \bullet Q))$

One point rules, for FOPLN we require the premise $\delta(E)$
23. $(\forall x \bullet x = E \Rightarrow P) \Leftrightarrow P[E/x]$
24. $(\exists x \bullet x = E \wedge P) \Leftrightarrow P[E/x]$

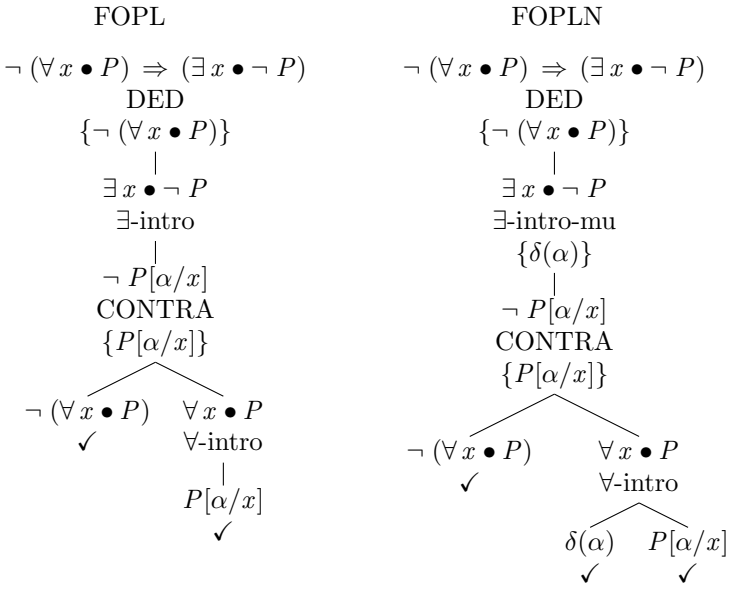
[1] P admits y for x if every free occurrence of x in P becomes a free occurrence of y in $P[y/x]$.

5 Logic toolbox proofs in both logics

5.1 de Morgan rules

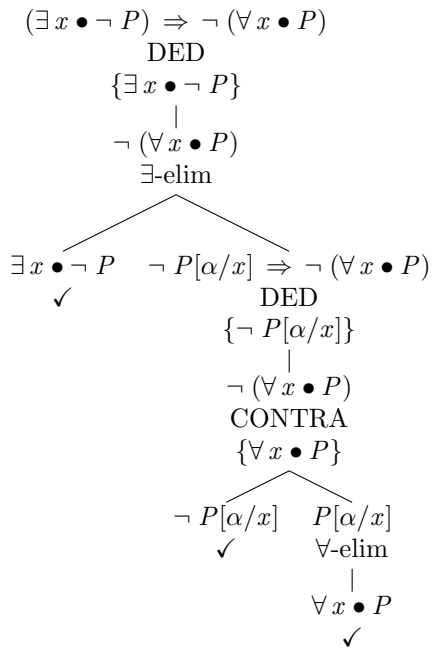
5.1.1 $\neg\,(\forall\, x \bullet P) \;\Leftrightarrow\; (\exists\, x \bullet \neg\, P)$

LHS $\Rightarrow$ RHS

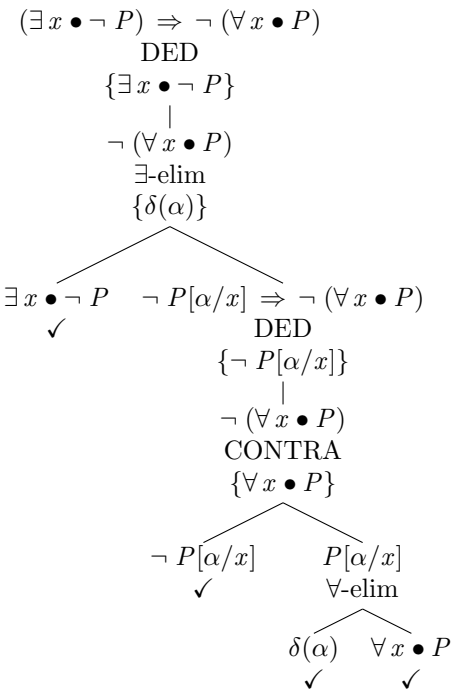


RHS $\Rightarrow$ LHS

FOPL



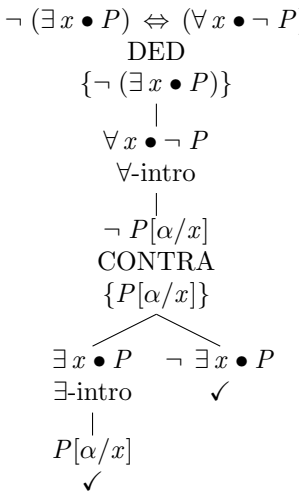
FOPLN



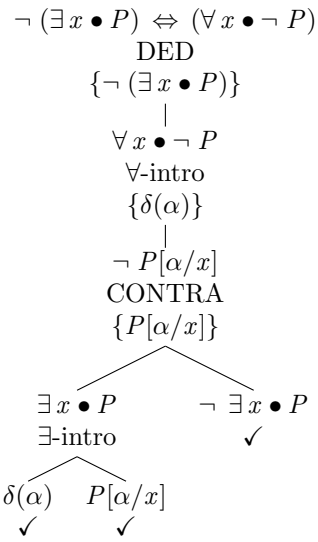
5.1.2 $\neg\,(\exists\, x \bullet P) \;\Leftrightarrow\; (\forall\, x \bullet \neg\, P)$

LHS $\Rightarrow$ RHS

FOPL

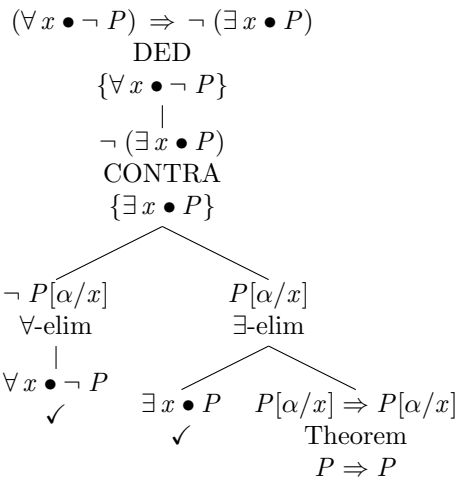


FOPLN

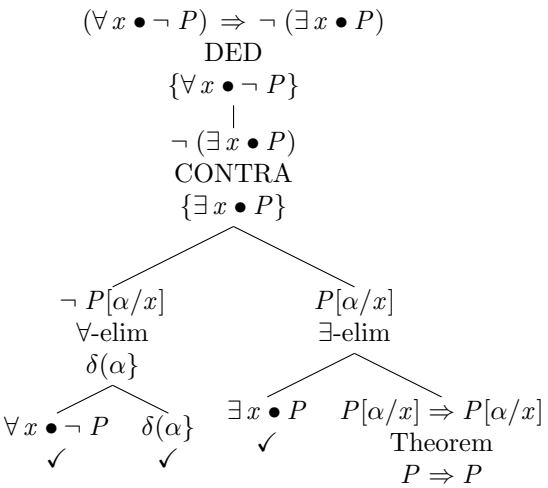


RHS $\Rightarrow$ LHS

FOPL



FOPLN



5.2 Change of order of bound variables

5.2.1 $(\forall\, x \bullet \forall\, y \bullet P) \;\Rightarrow\; (\forall\, y \bullet \forall\, x \bullet P)$

FOPL

$(\forall\, x \bullet \forall\, y \bullet P) \;\Rightarrow\; (\forall\, y \bullet \forall\, x \bullet P)$
DED
$\{\forall\, x \bullet \forall\, y \bullet P\}$
|
$\forall\, y \bullet \forall\, x \bullet P$
$\forall$-intro
|
$\forall\, x \bullet P[\alpha/y]$
$\forall$-intro
|
$P[\alpha/y][\alpha'/x]$
i.e.
$P[\alpha'/x][\alpha/y]$
$\forall$-elim
|
$\forall\, y \bullet P[\alpha'/x]$
$\forall$-elim
|
$\forall\, x \bullet \forall\, y \bullet P$
✓

FOPLN

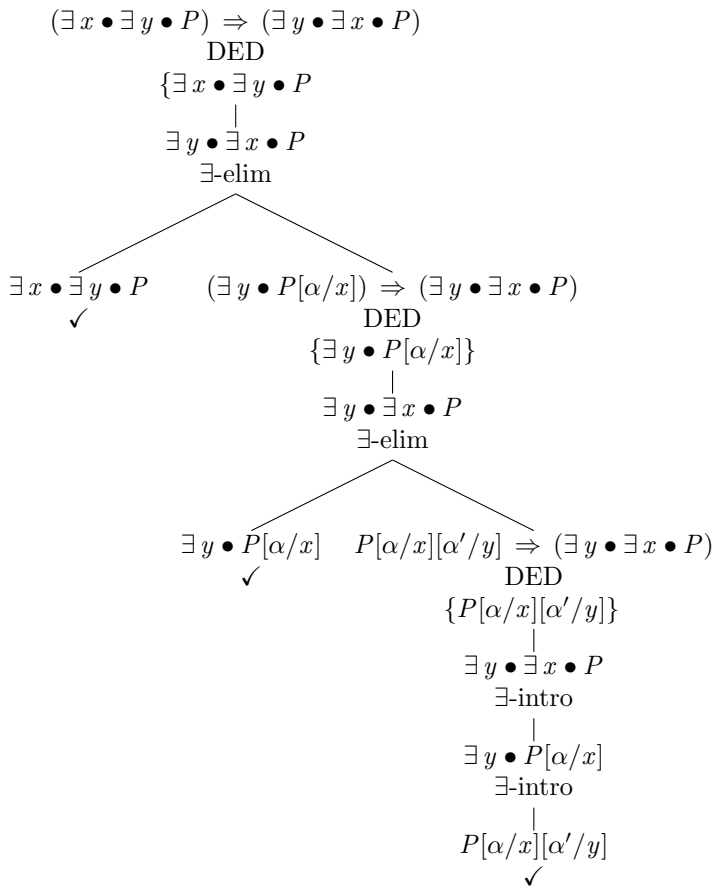
$(\forall\, x \bullet \forall\, y \bullet P) \;\Rightarrow\; (\forall\, y \bullet \forall\, x \bullet P)$
DED
$\{\forall\, x \bullet \forall\, y \bullet P\}$
|
$\forall\, y \bullet \forall\, x \bullet P$
$\forall$-intro
$\{\delta(\alpha)\}$
|
$\forall\, x \bullet P[\alpha/y]$
$\forall$-intro
$\{\delta(\alpha')\}$
|
$P[\alpha/y][\alpha'/x]$
i.e.
$P[\alpha'/x][\alpha/y]$
$\forall$-elim

$\delta(\alpha')$ ✓ — $\forall\, y \bullet P[\alpha'/x]$ $\forall$-elim

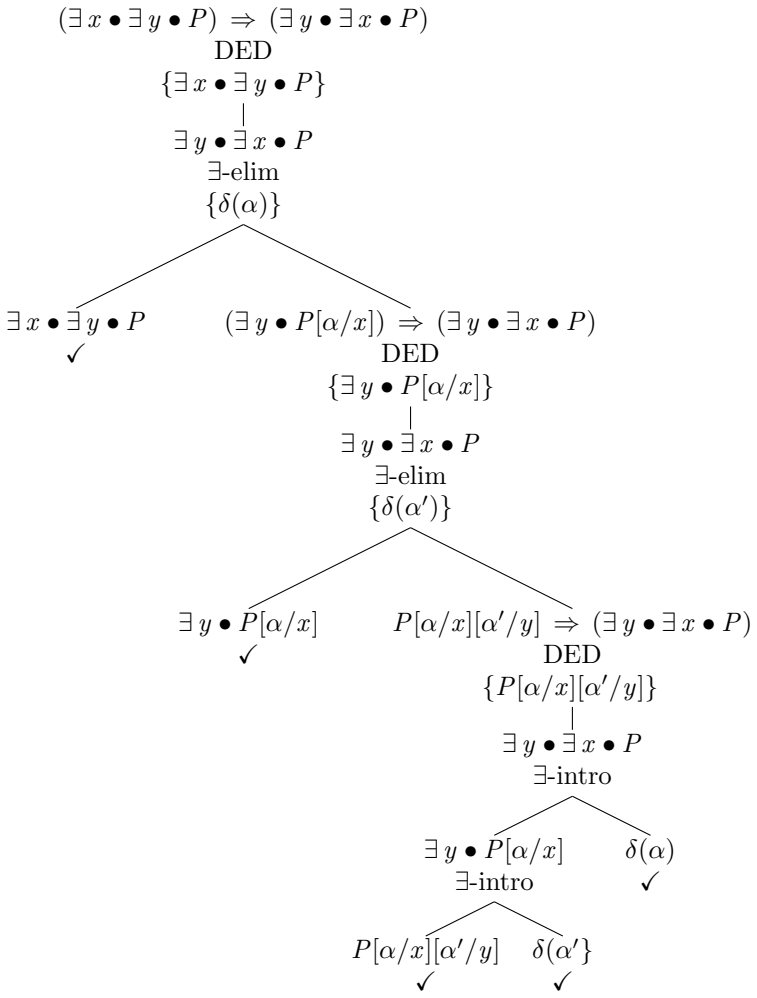
$\delta(\alpha')$ ✓ — $\forall\, x \bullet \forall\, y \bullet P$ ✓

5.2.2 $(\exists\, x \bullet \exists\, y \bullet P) \Rightarrow (\exists\, y \bullet \exists\, x \bullet P)$

FOPL



FOPLN

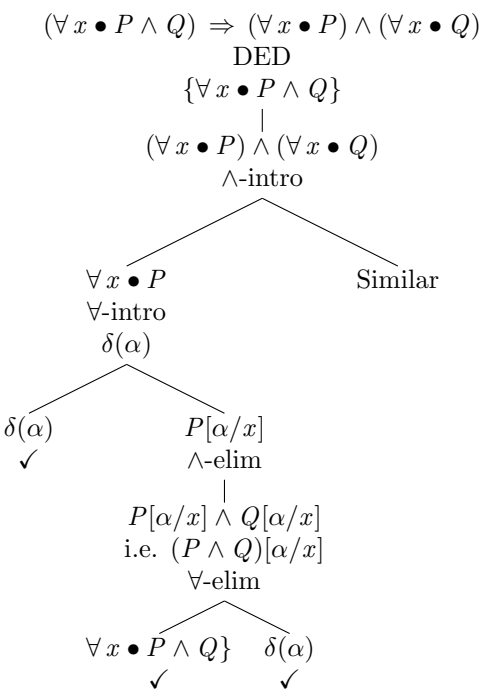

$(\exists\, x \bullet \exists\, y \bullet P) \Rightarrow (\exists\, y \bullet \exists\, x \bullet P)$
DED
$\{\exists\, x \bullet \exists\, y \bullet P\}$
$\exists\, y \bullet \exists\, x \bullet P$
∃-elim
$\{\delta(\alpha)\}$
$\exists\, x \bullet \exists\, y \bullet P$
✓
$(\exists\, y \bullet P[\alpha/x]) \Rightarrow (\exists\, y \bullet \exists\, x \bullet P)$
DED
$\{\exists\, y \bullet P[\alpha/x]\}$
$\exists\, y \bullet \exists\, x \bullet P$
∃-elim
$\{\delta(\alpha')\}$
$\exists\, y \bullet P[\alpha/x]$
✓
$P[\alpha/x][\alpha'/y] \Rightarrow (\exists\, y \bullet \exists\, x \bullet P)$
DED
$\{P[\alpha/x][\alpha'/y]\}$
$\exists\, y \bullet \exists\, x \bullet P$
∃-intro
$\exists\, y \bullet P[\alpha/x]$
∃-intro
$\delta(\alpha)$
✓
$P[\alpha/x][\alpha'/y]$
✓
$\delta(\alpha')$
✓

5.3 Splitting

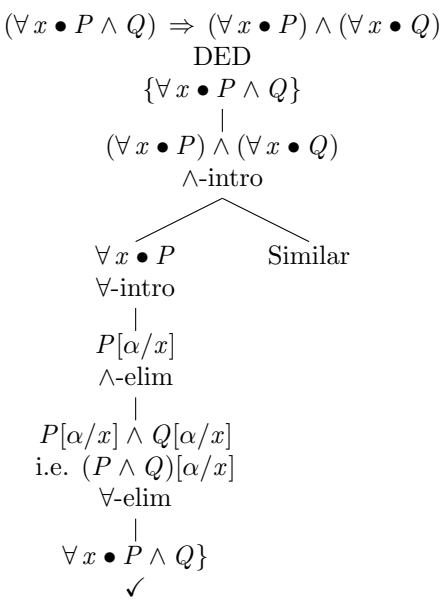
5.3.1 $(\forall\, x \bullet P \wedge Q) \Leftrightarrow (\forall\, x \bullet P) \wedge (\forall\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL



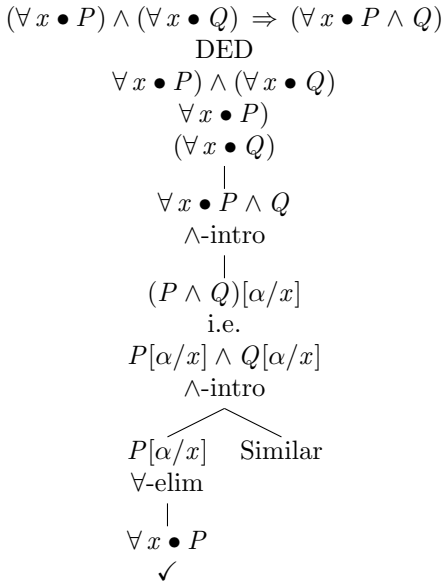
FOPLN

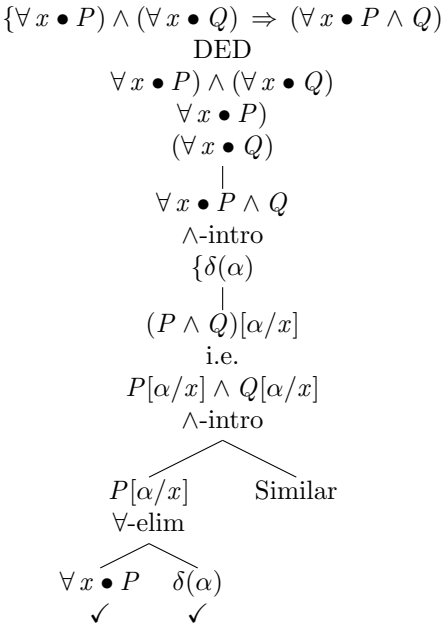


RHS $\Rightarrow$ LHS

FOPL

FOPLN

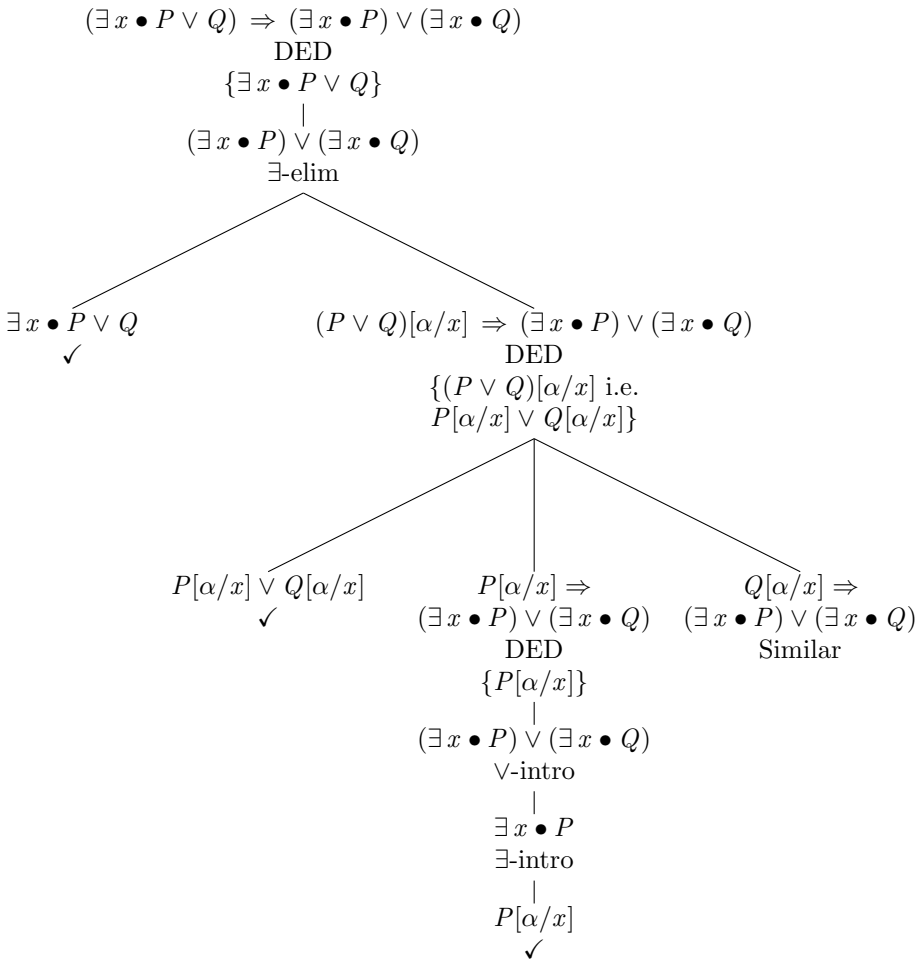

$(\forall\, x \bullet P) \wedge (\forall\, x \bullet Q) \;\Rightarrow\; (\forall\, x \bullet P \wedge Q)$
DED
$\forall\, x \bullet P) \wedge (\forall\, x \bullet Q)$
$\forall\, x \bullet P)$
$(\forall\, x \bullet Q)$
$\forall\, x \bullet P \wedge Q$
$\wedge$-intro
$(P \wedge Q)[\alpha/x]$
i.e.
$P[\alpha/x] \wedge Q[\alpha/x]$
$\wedge$-intro
$P[\alpha/x]$
Similar
$\forall$-elim
$\forall\, x \bullet P$
✓


$\{\forall\, x \bullet P) \wedge (\forall\, x \bullet Q) \;\Rightarrow\; (\forall\, x \bullet P \wedge Q)$
DED
$\forall\, x \bullet P) \wedge (\forall\, x \bullet Q)$
$\forall\, x \bullet P)$
$(\forall\, x \bullet Q)$
$\forall\, x \bullet P \wedge Q$
$\wedge$-intro
$\{\delta(\alpha)$
$(P \wedge Q)[\alpha/x]$
i.e.
$P[\alpha/x] \wedge Q[\alpha/x]$
$\wedge$-intro
$P[\alpha/x]$
Similar
$\forall$-elim
$\forall\, x \bullet P$
$\delta(\alpha)$
✓
✓

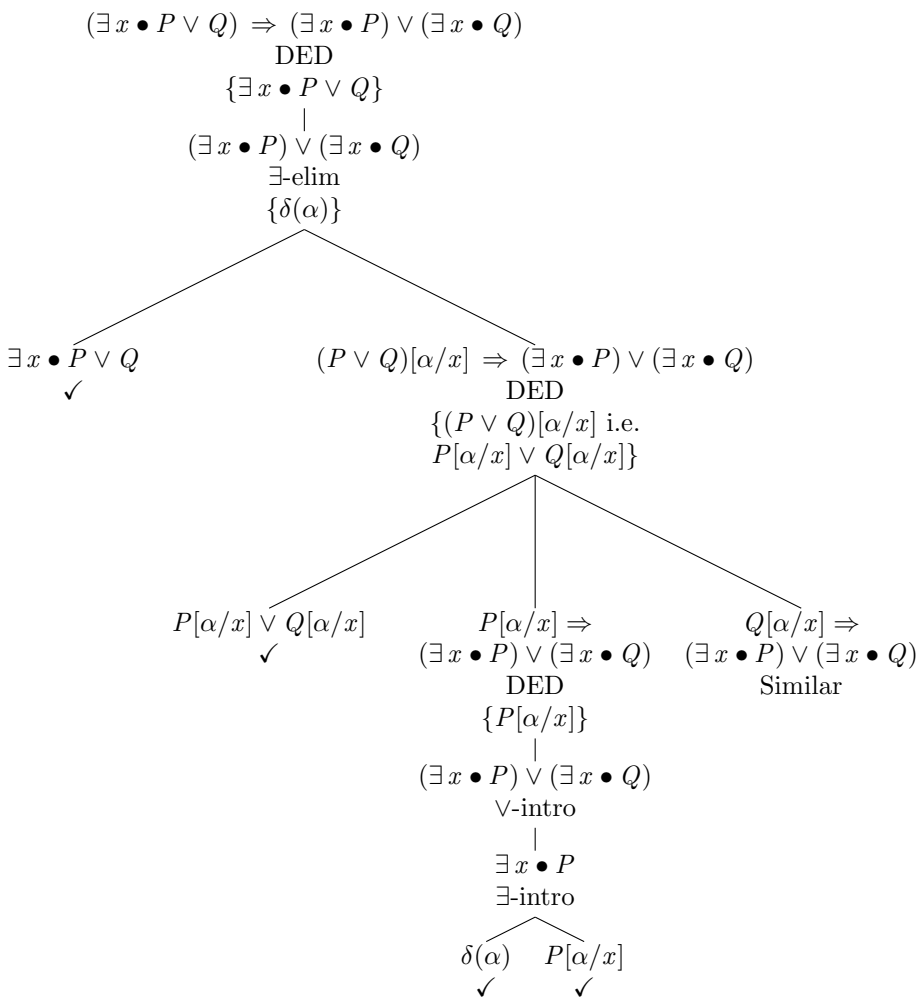
5.3.2 $(\exists\, x \bullet P \lor Q) \Leftrightarrow (\exists\, x \bullet P) \lor (\exists\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL

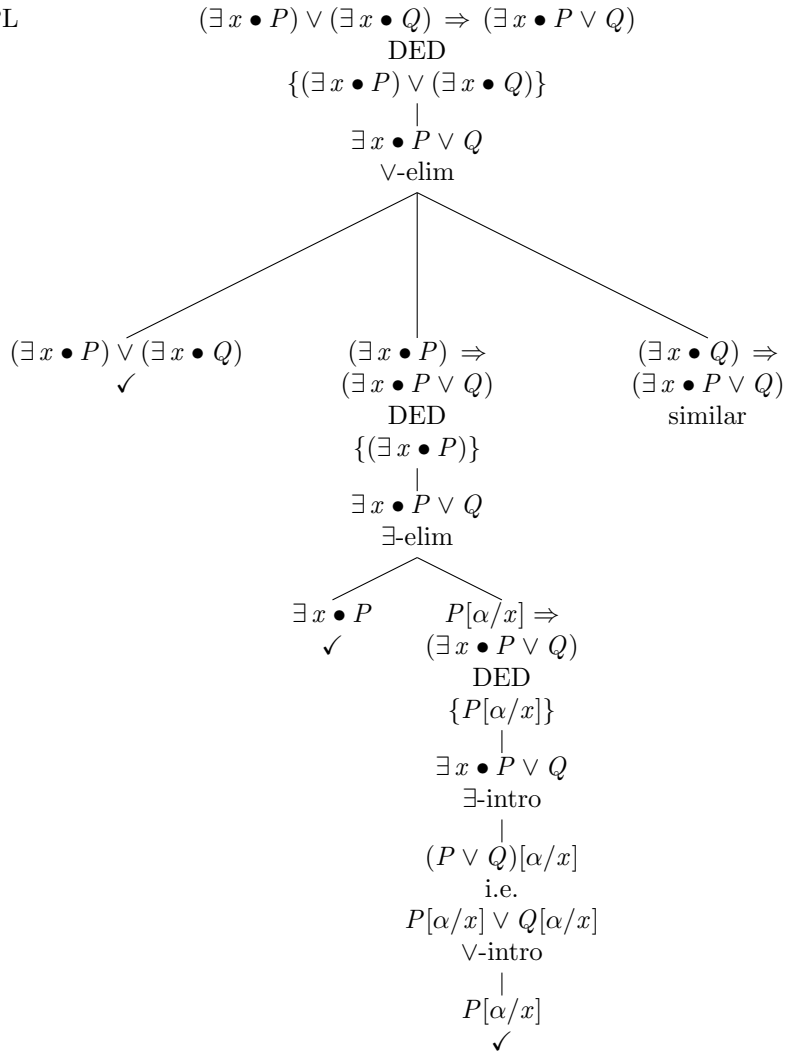


FOPLN

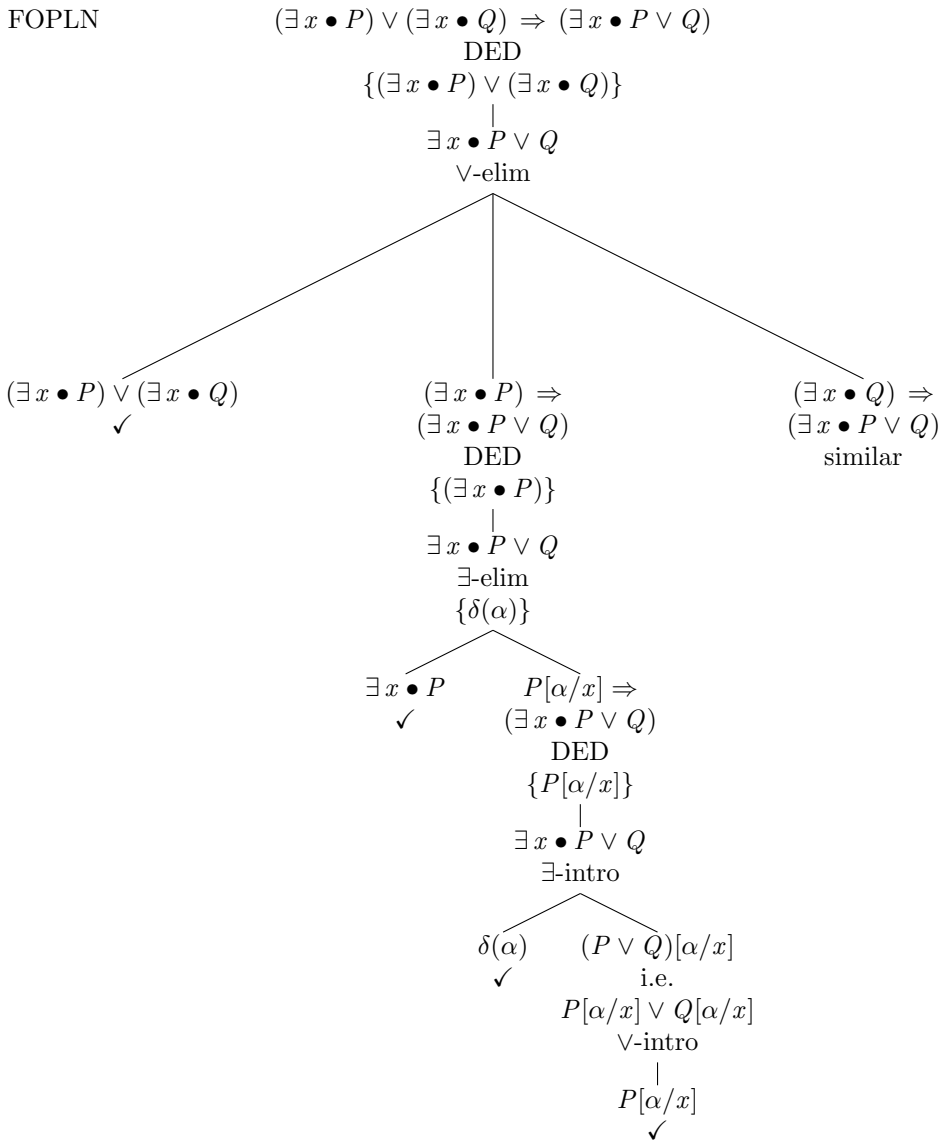

$(\exists\, x \bullet P \vee Q) \Rightarrow (\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
DED
$\{\exists\, x \bullet P \vee Q\}$
$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
∃-elim
$\{\delta(\alpha)\}$
$\exists\, x \bullet P \vee Q$
✓
$(P \vee Q)[\alpha/x] \Rightarrow (\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
DED
$\{(P \vee Q)[\alpha/x]$ i.e.
$P[\alpha/x] \vee Q[\alpha/x]\}$
$P[\alpha/x] \vee Q[\alpha/x]$
✓
$P[\alpha/x] \Rightarrow$
$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
DED
$\{P[\alpha/x]\}$
$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
∨-intro
$\exists\, x \bullet P$
∃-intro
$\delta(\alpha)$
✓
$P[\alpha/x]$
✓
$Q[\alpha/x] \Rightarrow$
$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
Similar

RHS $\Rightarrow$ LHS

FOPL

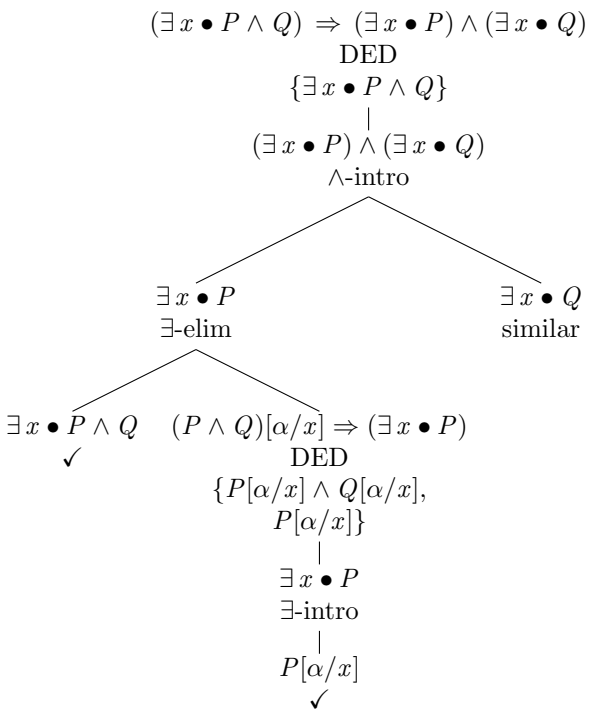


FOPLN

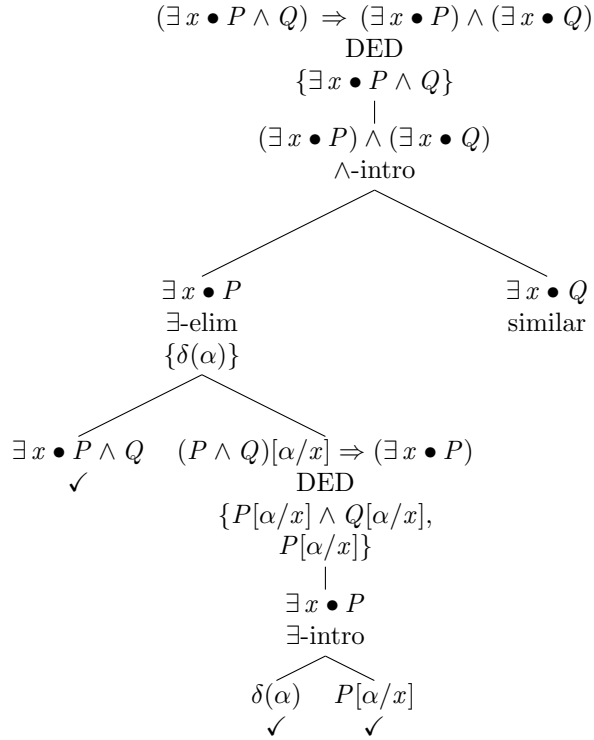

$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q) \;\Rightarrow\; (\exists\, x \bullet P \vee Q)$
DED
$\{(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)\}$
$\exists\, x \bullet P \vee Q$
$\vee$-elim
$(\exists\, x \bullet P) \vee (\exists\, x \bullet Q)$
✓
$(\exists\, x \bullet P) \Rightarrow$
$(\exists\, x \bullet P \vee Q)$
DED
$\{(\exists\, x \bullet P)\}$
$\exists\, x \bullet P \vee Q$
$\exists$-elim
$\{\delta(\alpha)\}$
$\exists\, x \bullet P$
✓
$P[\alpha/x] \Rightarrow$
$(\exists\, x \bullet P \vee Q)$
DED
$\{P[\alpha/x]\}$
$\exists\, x \bullet P \vee Q$
$\exists$-intro
$\delta(\alpha)$
✓
$(P \vee Q)[\alpha/x]$
i.e.
$P[\alpha/x] \vee Q[\alpha/x]$
$\vee$-intro
$P[\alpha/x]$
✓
$(\exists\, x \bullet Q) \Rightarrow$
$(\exists\, x \bullet P \vee Q)$
similar

5.3.3 $(\exists\, x \bullet P \wedge Q) \;\Rightarrow\; (\exists\, x \bullet P) \wedge (\exists\, x \bullet Q)$

FOPL



FOPLN



5.4 Where $x \setminus Q$

5.4.1 $Q \Leftrightarrow (\forall\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL

$Q \Rightarrow (\forall\, x \bullet Q)$
DED
$\{Q\}$
|
$\forall\, x \bullet Q$
$\forall$-intro
|
$Q[\alpha/x]$
i.e. since $x \setminus Q$
Q
✓

FOPLN

$Q \Rightarrow (\forall\, x \bullet Q)$
DED
$\{Q\}$
|
$\forall\, x \bullet Q$
$\forall$-intro
$\{\delta(\alpha)\}$
|
$Q[\alpha/x]$
i.e. since $x \setminus Q$
Q
✓

RHS $\Rightarrow$ LHS

FOPL

$(\forall\, x \bullet Q) \Rightarrow Q$
DED
$\{\forall\, x \bullet Q\}$
|
Q
i.e. since $x \setminus Q$
$Q[\alpha/x]$
$\forall$-elim
|
$\forall\, x \bullet Q$
✓

FOPLN

$(\forall\, x \bullet Q) \Rightarrow Q$
DED
$\{\forall\, x \bullet Q\}$
|
Q
i.e. since $x \setminus Q$
$Q[\alpha/x]$
$\forall$-elim
$\{\delta(\alpha)\}$

$\delta(\alpha)$ ✓ $\quad$ $\forall\, x \bullet Q$ ✓

5.4.2 $Q \Leftrightarrow (\exists\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL

$Q \Rightarrow (\exists\, x \bullet Q)$
DED
$\{Q\}$
|
$\exists\, x \bullet Q$
$\exists$-intro
|
$Q[E/x]$
i.e. since $x \setminus Q$
Q
✓

FOPLN

$Q \Rightarrow (\exists\, x \bullet Q)$
DED
$\{Q\}$
|
$\exists\, x \bullet Q$
$\exists$-intro-mu
$\delta(\alpha)$

$\delta(\alpha)$	$Q[\alpha/x]$
✓	i.e. since $x \setminus Q$
	Q
	✓

RHS $\Rightarrow$ LHS

FOPL

$(\exists\, x \bullet Q) \Rightarrow Q$
DED
$\{\exists\, x \bullet Q\}$
|
Q
$\exists$-elim

$\exists\, x \bullet Q$	$Q[E/x] \Rightarrow Q$
ok	i.e. since $x \setminus Q$
	$Q \Rightarrow Q$
	DED
	$\{Q\}$
	|
	Q
	✓

FOPLN

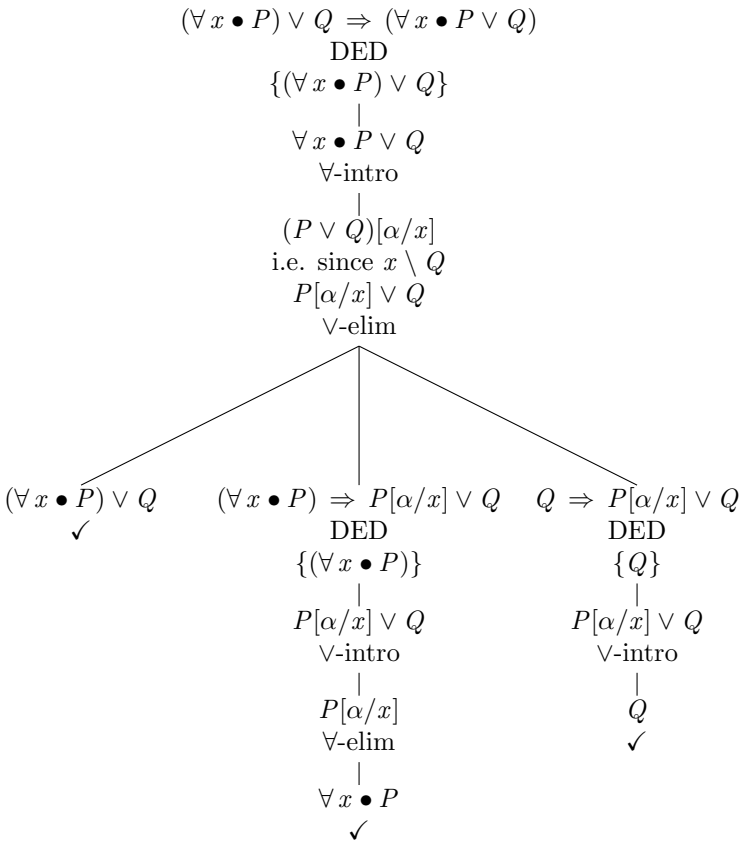
$(\exists\, x \bullet Q) \Rightarrow Q$
DED
$\{\exists\, x \bullet Q\}$
|
Q
$\exists$-elim
$\{\delta(\alpha)\}$

$\exists\, x \bullet Q$	$Q[\alpha/x] \Rightarrow Q$
ok	i.e. since $x \setminus Q$
	$Q \Rightarrow Q$
	DED
	$\{Q\}$
	|
	Q
	✓

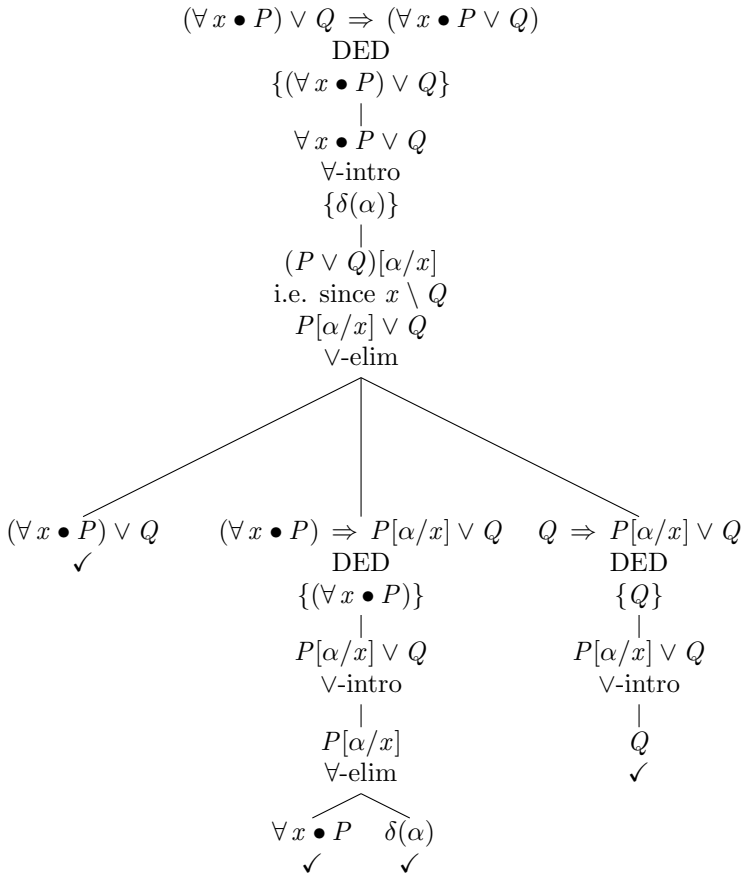
5.4.3 $(\forall\, x \bullet P) \vee Q \;\Leftrightarrow\; (\forall\, x \bullet P \vee Q)$

LHS $\Rightarrow$ RHS

FOPL

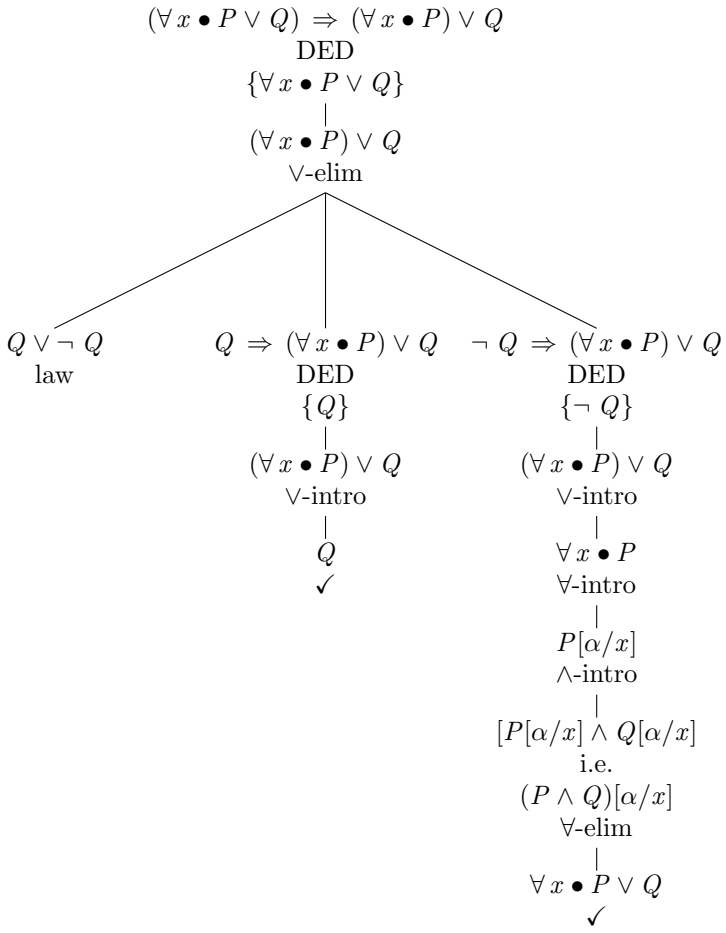


FOPLN

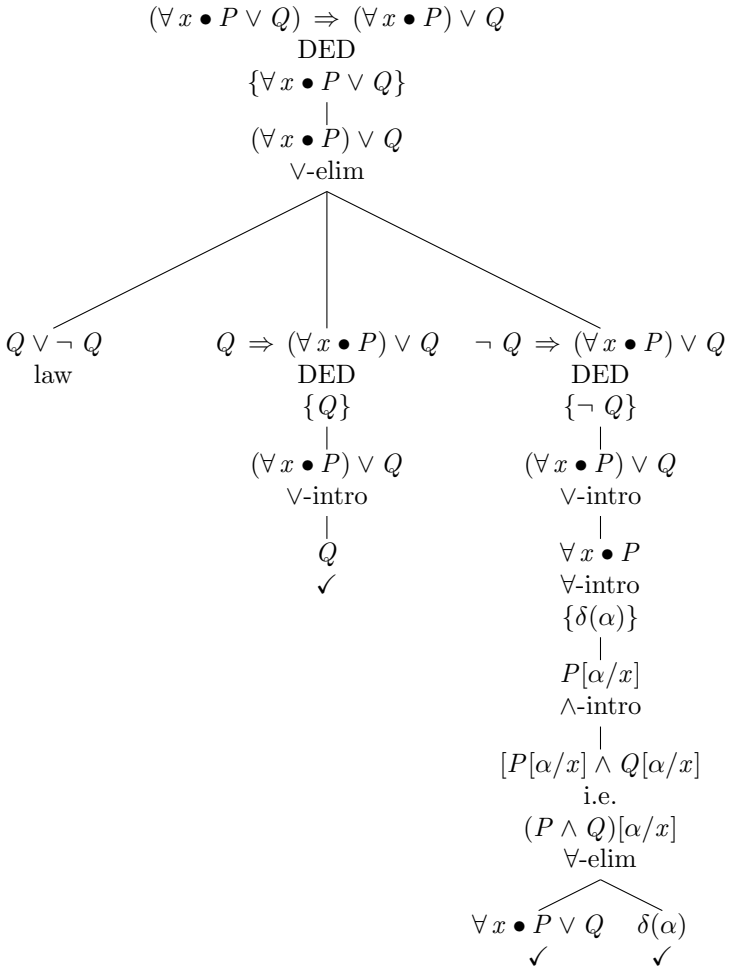

$(\forall\, x \bullet P) \vee Q \;\Rightarrow\; (\forall\, x \bullet P \vee Q)$
DED
$\{(\forall\, x \bullet P) \vee Q\}$
$\forall\, x \bullet P \vee Q$
∀-intro
$\{\delta(\alpha)\}$
$(P \vee Q)[\alpha/x]$
i.e. since $x \setminus Q$
$P[\alpha/x] \vee Q$
∨-elim
$(\forall\, x \bullet P) \vee Q$
✓
$(\forall\, x \bullet P) \;\Rightarrow\; P[\alpha/x] \vee Q$
DED
$\{(\forall\, x \bullet P)\}$
$P[\alpha/x] \vee Q$
∨-intro
$P[\alpha/x]$
∀-elim
$\forall\, x \bullet P$
✓
$\delta(\alpha)$
✓
$Q \;\Rightarrow\; P[\alpha/x] \vee Q$
DED
$\{Q\}$
$P[\alpha/x] \vee Q$
∨-intro
Q
✓

RHS $\Rightarrow$ LHS

FOPL



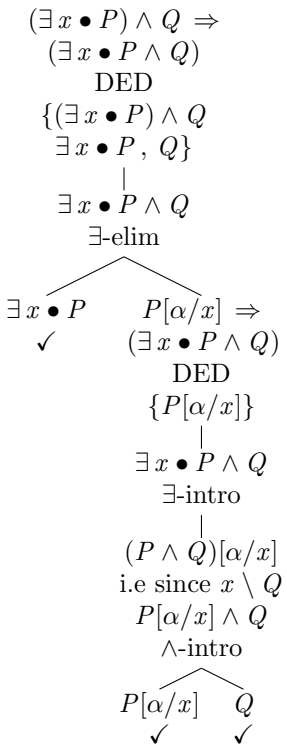
FOPLN


$(\forall\, x \bullet P \vee Q) \Rightarrow (\forall\, x \bullet P) \vee Q$
DED
$\{\forall\, x \bullet P \vee Q\}$
$(\forall\, x \bullet P) \vee Q$
$\vee$-elim
$Q \vee \neg\, Q$
law
$Q \Rightarrow (\forall\, x \bullet P) \vee Q$
DED
$\{Q\}$
$(\forall\, x \bullet P) \vee Q$
$\vee$-intro
Q
✓
$\neg\, Q \Rightarrow (\forall\, x \bullet P) \vee Q$
DED
$\{\neg\, Q\}$
$(\forall\, x \bullet P) \vee Q$
$\vee$-intro
$\forall\, x \bullet P$
$\forall$-intro
$\{\delta(\alpha)\}$
$P[\alpha/x]$
$\wedge$-intro
$[P[\alpha/x] \wedge Q[\alpha/x]$
i.e.
$(P \wedge Q)[\alpha/x]$
$\forall$-elim
$\forall\, x \bullet P \vee Q$
✓
$\delta(\alpha)$
✓

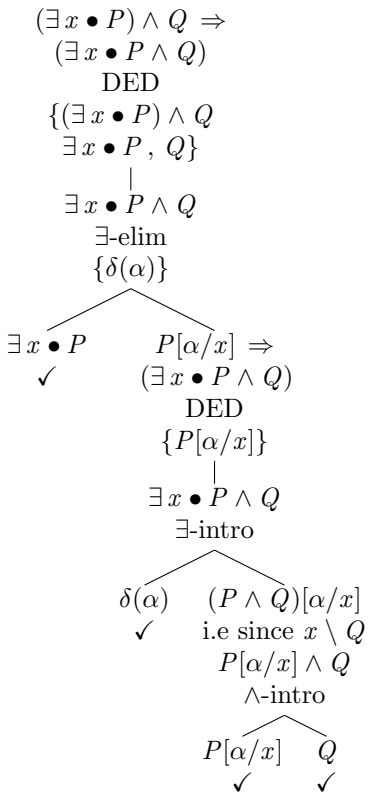
5.4.4 $(\exists\, x \bullet P) \wedge Q \;\Leftrightarrow\; (\exists\, x \bullet P \wedge Q)$

LHS $\Rightarrow$ RHS

FOPL

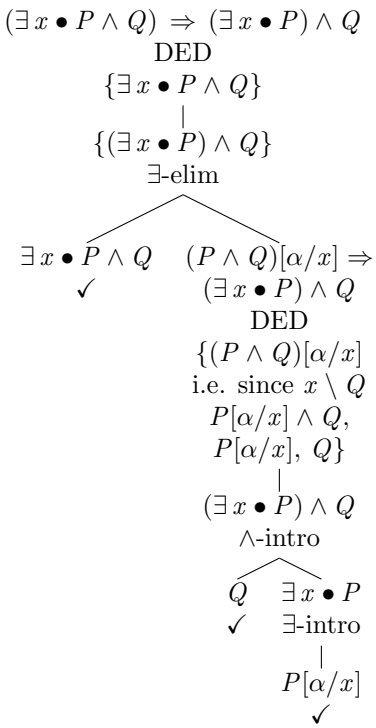


FOPLN

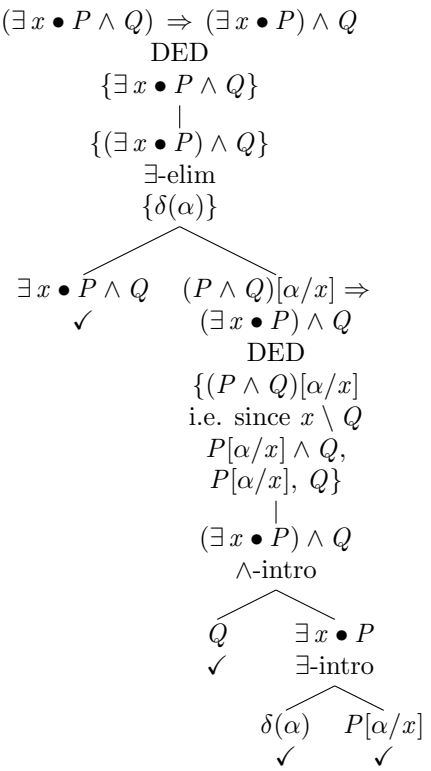


RHS $\Rightarrow$ LHS

FOPL



FOPLN



5.4.5 $(\forall\, x \bullet Q \Rightarrow P) \;\Leftrightarrow\; Q \Rightarrow (\forall\, x \bullet P)$

LHS $\Rightarrow$ RHS FOPL

$(\forall\, x \bullet Q \Rightarrow P) \;\Rightarrow\; (Q \Rightarrow (\forall\, x \bullet P))$

DED
$\{\forall\, x \bullet Q \Rightarrow P\}$
|
$Q \Rightarrow (\forall\, x \bullet P)$
DED
$\{Q\}$
|
$\forall\, x \bullet P$
$\forall$-intro
|
$P[\alpha/x]$
MP

Q ✓ — $Q \Rightarrow P[\alpha/x]$ i.e. since $x \setminus Q$ $(Q \Rightarrow P)[\alpha/x]$ $\forall$-elim
|
$\forall\, x \bullet Q \Rightarrow P$
✓

FOPLN

$(\forall\, x \bullet Q \Rightarrow P) \;\Rightarrow\; (Q \Rightarrow (\forall\, x \bullet P))$

DED
$\{\forall\, x \bullet Q \Rightarrow P\}$
|
$Q \Rightarrow (\forall\, x \bullet P)$
DED
$\{Q\}$
|
$\forall\, x \bullet P$
$\forall$-intro
$\{\delta(\alpha)\}$
|
$P[\alpha/x]$
MP

Q ✓ — $Q \Rightarrow P[\alpha/x]$ i.e. since $x \setminus Q$ $(Q \Rightarrow P)[\alpha/x]$ $\forall$-elim

$\delta(\alpha)$ ✓ — $\forall\, x \bullet Q \Rightarrow P$ ✓

RHS $\Rightarrow$ LHS

$(Q \Rightarrow (\forall\, x \bullet P)) \;\Rightarrow\; (\forall\, x \bullet Q \Rightarrow P)$

DED
$\{Q \Rightarrow (\forall\, x \bullet P)\}$
|
$\forall\, x \bullet Q \Rightarrow P$
$\forall$-intro
|
$(Q \Rightarrow P)[\alpha/x]$
i.e. since $x \setminus Q$
$Q \Rightarrow P[\alpha/x]$
DED
$\{Q\}$
|
$P[\alpha/x]$
$\forall$-elim
|
$\forall\, x \bullet P$
MP

Q ✓ — $Q \Rightarrow (\forall\, x \bullet P)$ ✓

$(Q \Rightarrow (\forall\, x \bullet P)) \;\Rightarrow\; (\forall\, x \bullet Q \Rightarrow P)$

DED
$\{Q \Rightarrow (\forall\, x \bullet P)\}$
|
$\forall\, x \bullet Q \Rightarrow P$
$\forall$-intro
$\{\delta(\alpha)\}$
|
$(Q \Rightarrow P)[\alpha/x]$
i.e. since $x \setminus Q$
$Q \Rightarrow P[\alpha/x]$
DED
$\{Q\}$
|
$P[\alpha/x]$
$\forall$-elim

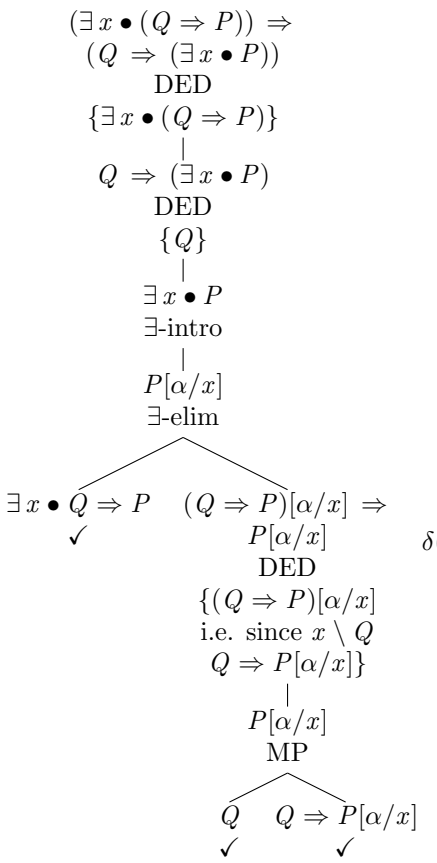
$\delta(\alpha)$ ✓ — $\forall\, x \bullet P$ MP

Q ✓ — $Q \Rightarrow (\forall\, x \bullet P)$ ✓

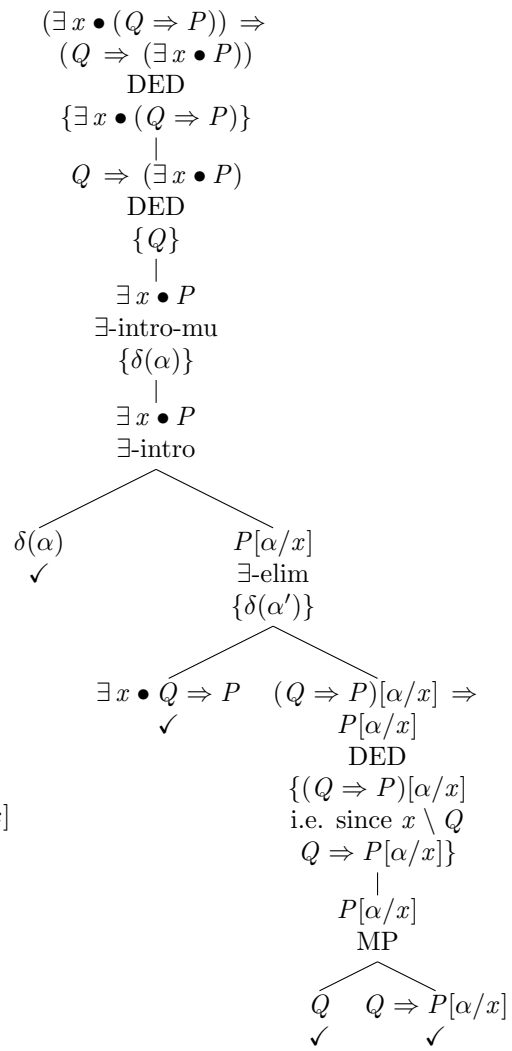
5.4.6 $(\exists\, x \bullet (Q \Rightarrow P)) \Leftrightarrow Q \Rightarrow (\exists\, x \bullet P)$

LHS $\Rightarrow$ RHS

FOPL



FOPLN

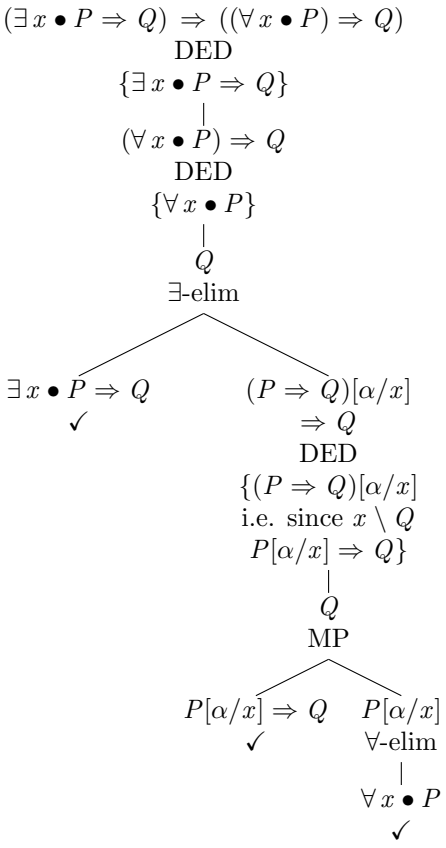


5.5 Changing scope and quantifiers where $x \setminus Q$

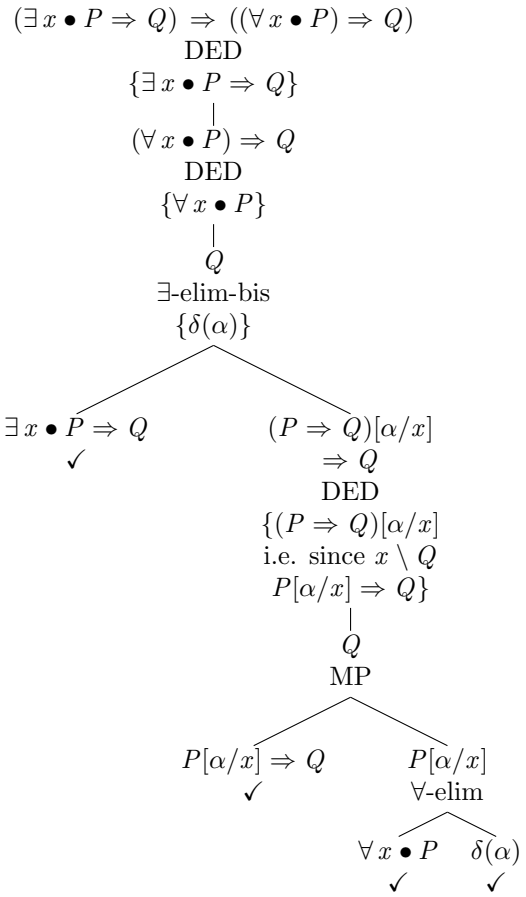
5.5.1 $(\exists\, x \bullet P \Rightarrow Q) \Leftrightarrow (\forall\, x \bullet P) \Rightarrow Q$

LHS $\Rightarrow$ RHS

FOPL



FOPLN



RHS $\Rightarrow$ LHS

FOPL

$((\forall\, x \bullet P) \Rightarrow Q) \;\Rightarrow\; (\exists\, x \bullet P \Rightarrow Q)$

DED

$\{(\forall\, x \bullet P) \Rightarrow Q\}$

$\exists\, x \bullet P \Rightarrow Q$

$\exists$-intro

$(P \Rightarrow Q)[\alpha/x]$

i.e. since $x \setminus Q$

$P[\alpha/x] \Rightarrow Q$

DED

$\{P[\alpha/x]\}$

Q

MP

$\{\forall\, x \bullet P) \Rightarrow Q$ ✓

$\forall\, x \bullet P$

$\forall$-intro

$P[\alpha/x]$

✓

FOPLN

$((\forall\, x \bullet P) \Rightarrow Q) \;\Rightarrow\; (\exists\, x \bullet P \Rightarrow Q)$

DED

$\{(\forall\, x \bullet P) \Rightarrow Q\}$

$\exists\, x \bullet P \Rightarrow Q$

$\exists$-intro-bis

$\{\delta(\alpha)\}$

$(P \Rightarrow Q)[\alpha/x]$

i.e. since $x \setminus Q$

$P[\alpha/x] \Rightarrow Q$

DED

$\{P[\alpha/x]\}$

Q

MP

$\{\forall\, x \bullet P) \Rightarrow Q$ ✓

$\forall\, x \bullet P$

$\forall$-intro

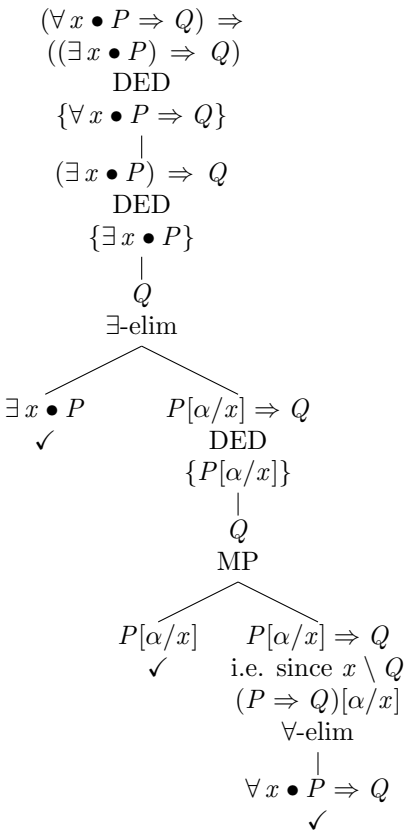
$\{\delta(\alpha')\}$

$P[\alpha/x]$ ✓ $\quad \delta(\alpha)$ ✓

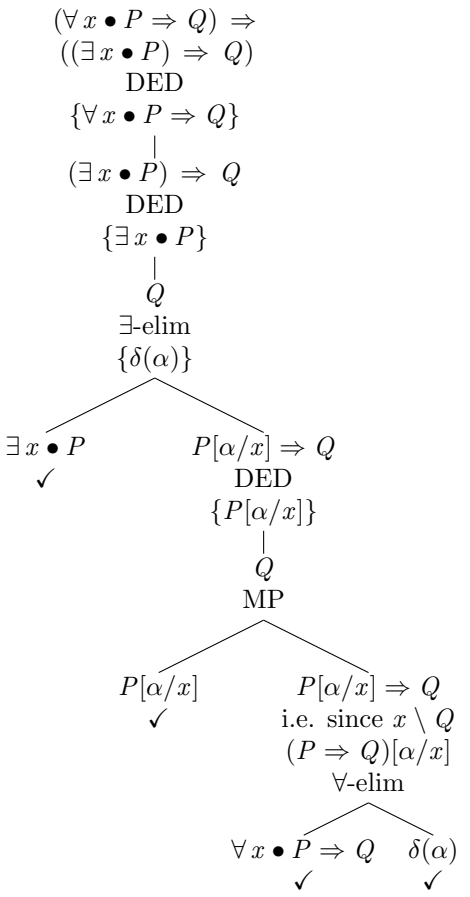
5.5.2 $(\forall\, x \bullet P \Rightarrow Q) \Leftrightarrow (\exists\, x \bullet P) \Rightarrow Q$

LHS $\Rightarrow$ RHS

FOPL



FOPLN



RHS $\Rightarrow$ LHS

FOPL

$((\exists\, x \bullet P) \Rightarrow Q) \;\Rightarrow\; (\forall\, x \bullet P \Rightarrow Q)$

DED

$\{(\exists\, x \bullet P) \Rightarrow Q)\}$

$\forall\, x \bullet P \Rightarrow Q$

$\forall$-intro

$(P \Rightarrow Q)[\alpha/x]$

i.e. since $x \setminus Q$

$P[\alpha/x] \Rightarrow Q$

DED

$\{P[\alpha/x]\}$

Q

MP

$(\exists\, x \bullet P) \Rightarrow Q$ ✓ $\qquad$ $\exists\, x \bullet P$

$\exists$-intro

$P[\alpha/x]$ ✓

FOPLN

$((\exists\, x \bullet P) \Rightarrow Q) \;\Rightarrow\; (\forall\, x \bullet P \Rightarrow Q)$

DED

$\{(\exists\, x \bullet P) \Rightarrow Q)\}$

$\forall\, x \bullet P \Rightarrow Q$

$\forall$-intro

$\{\delta(\alpha)\}$

$(P \Rightarrow Q)[\alpha/x]$

i.e. since $x \setminus Q$

$P[\alpha/x] \Rightarrow Q$

DED

$\{P[\alpha/x]\}$

Q

MP

$(\exists\, x \bullet P) \Rightarrow Q$ ✓ $\qquad$ $\exists\, x \bullet P$

$\exists$-intro

$\delta(\alpha)$ ✓ $\qquad$ $P[\alpha/x]$ ✓

5.6 Change of bound variable name

5.6.1 $(\forall\, x \bullet P) \Leftrightarrow (\forall\, y \bullet P[y/x])$

LHS $\Rightarrow$ RHS

FOPL

$$
\begin{array}{c}
(\forall\, x \bullet P) \Rightarrow (\forall\, y \bullet P[y/x]) \\
\text{DED} \\
\{\forall\, x \bullet P\} \\
| \\
\forall\, y \bullet P[y/x] \\
\forall\text{-intro} \\
| \\
P[y/x][\alpha/y] \\
\text{i.e.} \\
P[\alpha/x] \\
\forall\text{-elim} \\
| \\
\forall\, x \bullet P \\
\checkmark
\end{array}
$$

FOPLN

$$
\begin{array}{c}
(\forall\, x \bullet P) \Rightarrow (\forall\, y \bullet P[y/x]) \\
\text{DED} \\
\{\forall\, x \bullet P\} \\
| \\
\forall\, y \bullet P[y/x] \\
\forall\text{-intro} \\
\{\delta(\alpha)\} \\
| \\
P[y/x][\alpha/y] \\
\text{i.e.} \\
P[\alpha/x] \\
\forall\text{-elim} \\
\begin{array}{cc}
\delta(\alpha) & \forall\, x \bullet P \\
\checkmark & \checkmark
\end{array}
\end{array}
$$

RHS $\Rightarrow$ LHS

FOPL

$$
\begin{array}{c}
(\forall\, y \bullet P[y/x]) \Rightarrow (\forall\, x \bullet P) \\
\text{DED} \\
\{\forall\, y \bullet P[y/x]\} \\
| \\
\forall\, x \bullet P \\
\forall\text{-intro} \\
| \\
P[\alpha/x] \\
\text{i.e } P[y/x][\alpha/y] \\
\forall\text{-intro} \\
| \\
\forall\, y \bullet P[y/x] \\
\checkmark
\end{array}
$$

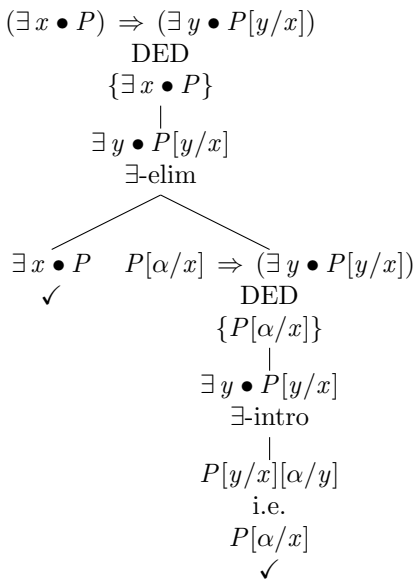
FOPLN

$$
\begin{array}{c}
(\forall\, y \bullet P[y/x]) \Rightarrow (\forall\, x \bullet P) \\
\text{DED} \\
\{\forall\, y \bullet P[y/x]\} \\
| \\
\forall\, x \bullet P \\
\forall\text{-intro} \\
\{\delta(\alpha)\} \\
| \\
P[\alpha/x] \\
\text{i.e } P[y/x][\alpha/y] \\
\forall\text{-intro} \\
\begin{array}{cc}
\delta(\alpha) & \forall\, y \bullet P[y/x] \\
\checkmark & \checkmark
\end{array}
\end{array}
$$

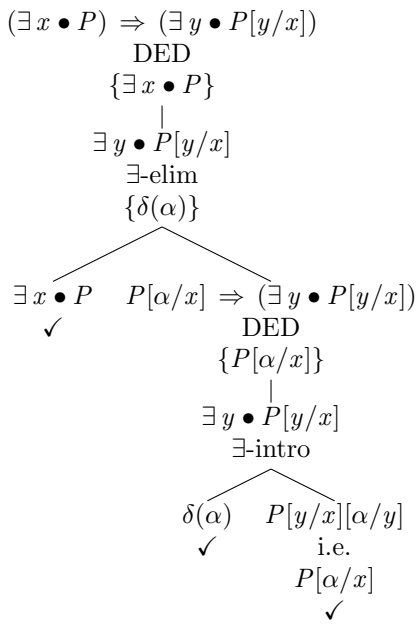
5.6.2 $(\exists\, x \bullet P) \Leftrightarrow (\exists\, y \bullet P[y/x])$

LHS $\Rightarrow$ RHS

FOPL

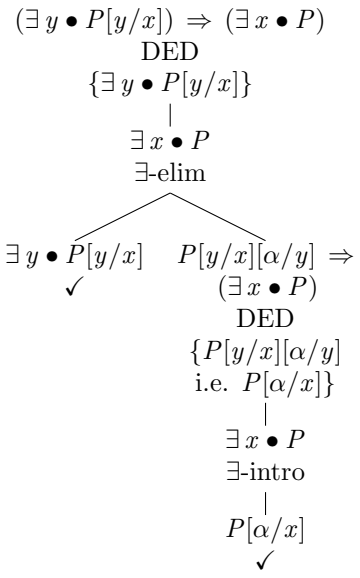


FOPLN

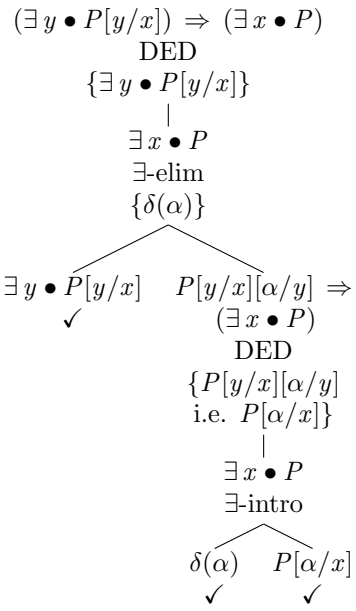


RHS $\Rightarrow$ LHS

FOPL



FOPLN



5.7 Monotonicity

5.7.1 $(\forall\, x \bullet P \Rightarrow Q) \Leftrightarrow (\forall\, x \bullet P) \Rightarrow (\forall\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL

$(\forall\, x \bullet P \Rightarrow Q) \Rightarrow$
$(\forall\, x \bullet P) \Rightarrow (\forall\, x \bullet Q)$
DED
$\{\forall\, x \bullet P \Rightarrow Q\}$
|
$(\forall\, x \bullet P) \Rightarrow (\forall\, x \bullet Q)$
DED
$\{\forall\, x \bullet P\}$
|
$\forall\, x \bullet Q$
$\forall$-intro
|
$Q[\alpha/x]$
MP

$P[\alpha/x] \Rightarrow Q[\alpha/x]$ — i.e. $(P \Rightarrow Q)[\alpha/x]$ — $\forall$-elim — $\forall\, x \bullet P \Rightarrow Q$ — ✓

$P[\alpha/x]$ — $\forall$-elim — $\forall\, x \bullet P$ — ✓

FOPLN

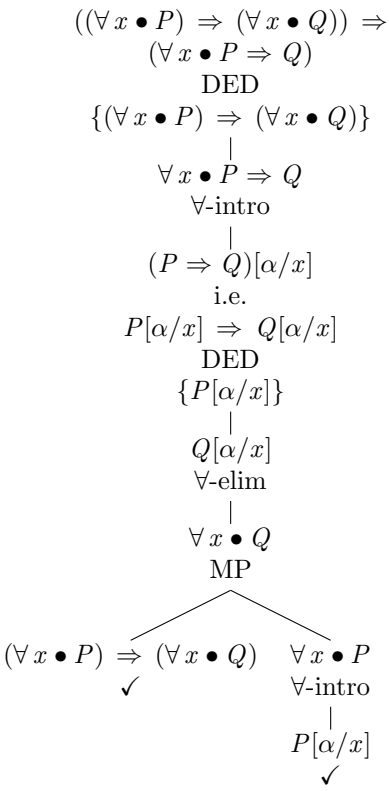
$(\forall\, x \bullet P \Rightarrow Q) \Rightarrow$
$(\forall\, x \bullet P) \Rightarrow (\forall\, x \bullet Q)$
DED
$\{\forall\, x \bullet P \Rightarrow Q\}$
|
$(\forall\, x \bullet P) \Rightarrow (\forall\, x \bullet Q)$
DED
$\{\forall\, x \bullet P\}$
|
$\forall\, x \bullet Q$
$\forall$-intro
$\{\delta(\alpha)\}$
|
$Q[\alpha/x]$
MP

$P[\alpha/x] \Rightarrow Q[\alpha/x]$ — i.e. $(P \Rightarrow Q)[\alpha/x]$ — $\forall$-elim — $\forall\, x \bullet P \Rightarrow Q$ — ✓

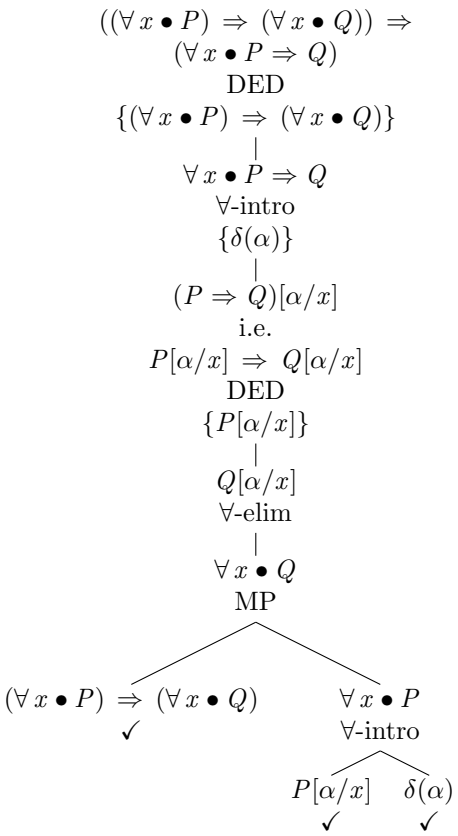
$P[\alpha/x]$ — $\forall$-elim — $\delta(\alpha)$ ✓ ; $\forall\, x \bullet P$ ✓

RHS $\Rightarrow$ LHS

FOPL



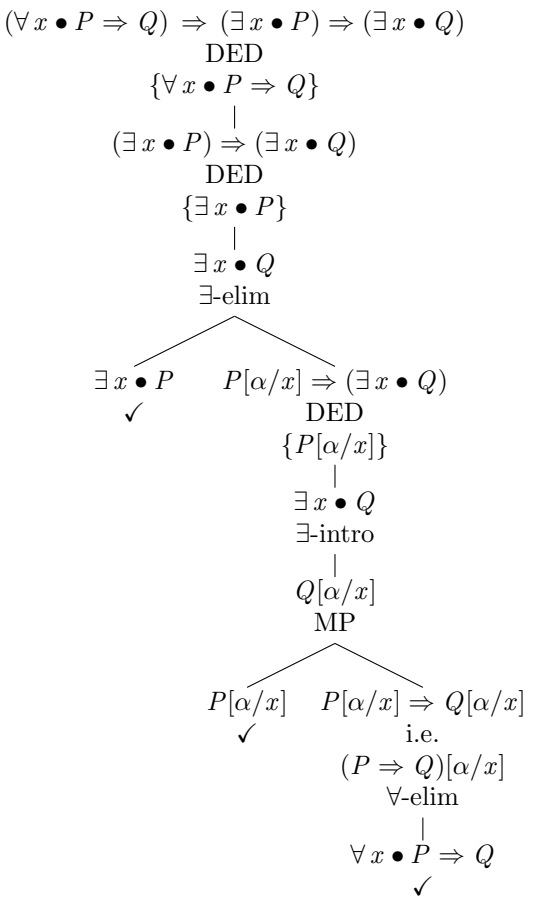
FOPLN



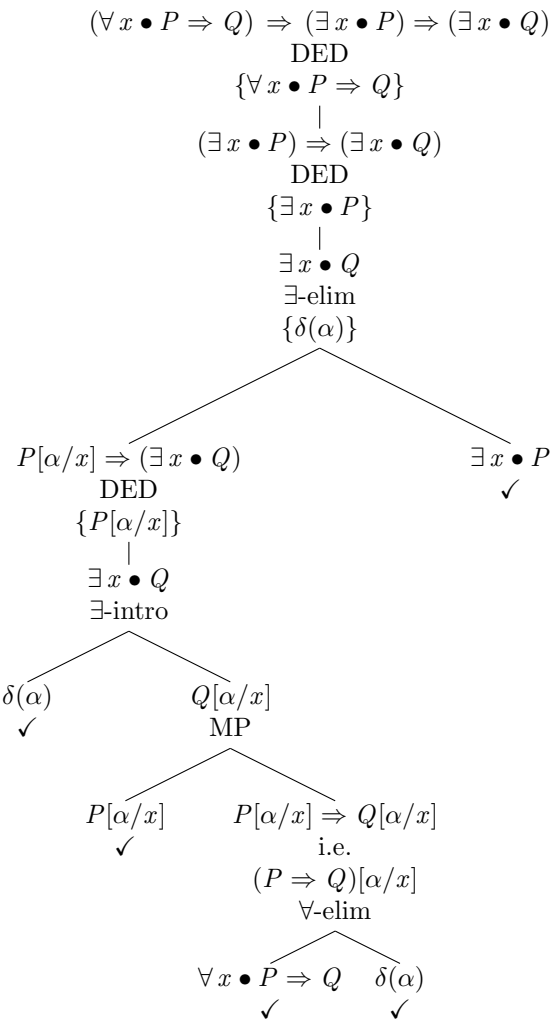
5.7.2 $(\forall\, x \bullet P \Rightarrow Q) \;\Rightarrow\; (\exists\, x \bullet P) \Rightarrow (\exists\, x \bullet Q)$

LHS $\Rightarrow$ RHS

FOPL



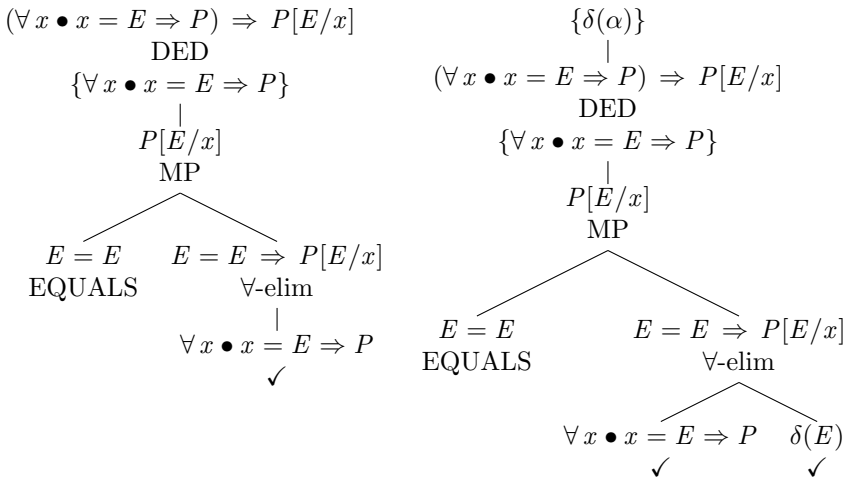
FOPLN



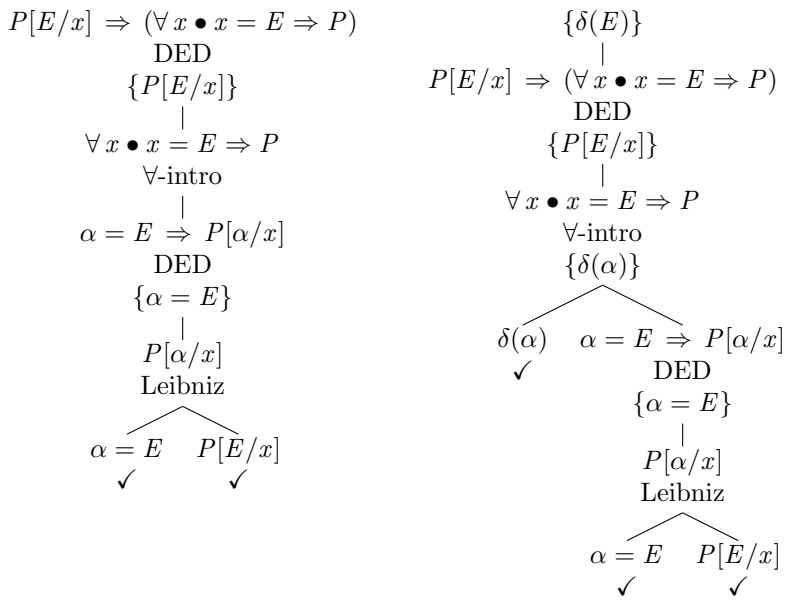
5.8 One point laws

5.8.1 $(\forall\, x \bullet x = E \Rightarrow P) \;\Leftrightarrow\; P[E/x]$ **(FOPL)**
$\delta(E) \vdash (\forall\, x \bullet x = E \Rightarrow P) \;\Leftrightarrow\; P[E/x]$ **(FOPLN)**

LHS $\Rightarrow$ RHS

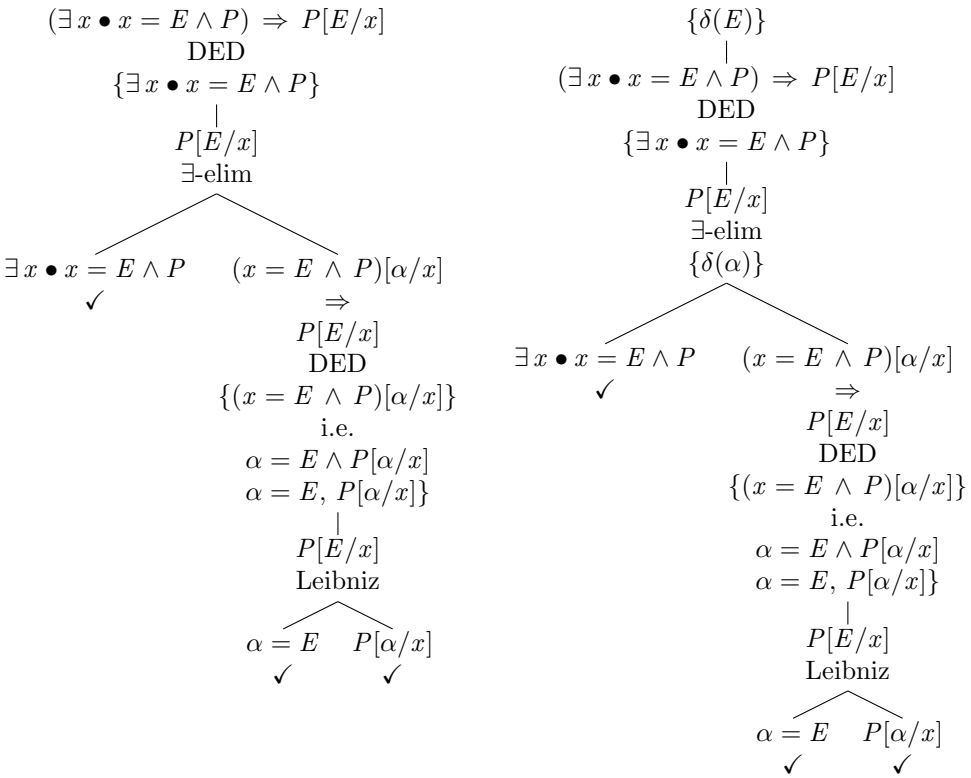


RHS $\Rightarrow$ LHS

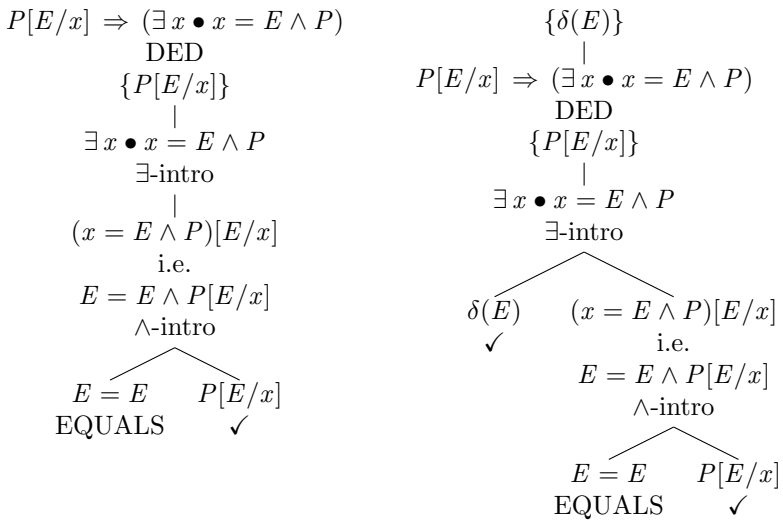


5.8.2 $(\exists\, x \bullet x = E \wedge P) \Leftrightarrow P[E/x]$ **(FOPL)**
$\delta(E) \vdash (\exists\, x \bullet x = E \wedge P) \Leftrightarrow P[E/x]$ **(FOPLN)**

LHS $\Rightarrow$ RHS



RHS $\Rightarrow$ LHS



Appendix

A Inference rules

A.1 Propositional logic proof rules

$$\frac{HYP \vdash P\,,\ HYP \vdash Q}{HYP \vdash P \wedge Q}\ \text{CONJ or } \wedge\text{-intro}$$

$$\frac{HYP \vdash P \wedge Q}{HYP \vdash P}\quad \frac{HYP \vdash P \wedge Q}{HYP \vdash Q}\ \text{AND or } \wedge\text{-elim}$$

$$\frac{HYP \vdash P\,,\ HYP \vdash P \Rightarrow Q}{HYP \vdash Q}\ \text{MODUS PONENS}$$

$$\frac{HYP \vdash P \Rightarrow Q,\quad HYP \vdash Q \Rightarrow P}{HYP \vdash P \Leftrightarrow Q}\ \text{EQUIV or } \Leftrightarrow\text{-intro}$$

$$\frac{HYP \vdash P \Leftrightarrow Q}{HYP \vdash P \Rightarrow Q}\quad \frac{HYP \vdash P \Leftrightarrow Q}{HYP \vdash Q \Rightarrow P}\ \text{IMPLIES or } \Leftrightarrow\text{-elim}$$

$$\frac{HYP \vdash P}{HYP \vdash P \vee Q}\quad \frac{HYP \vdash Q}{HYP \vdash P \vee Q}\ \text{OR or } \vee\text{-intro}$$

$$\frac{HYP \vdash P \Rightarrow R\quad HYP \vdash Q \Rightarrow R\quad HYP \vdash P \vee Q}{HYP \vdash R}\ \text{CASES or } \vee\text{-elim}$$

$$\frac{HYP,\ P \vdash Q}{HYP \vdash P \Rightarrow Q}\ \text{DED}\qquad \frac{}{HYP, P \vdash P}\ \text{INHYP}$$

$$\left.\begin{array}{c}\dfrac{HYP,\ P \vdash Q\,,\ HYP,\ P \vdash \neg Q}{HYP \vdash \neg P}\\[2ex] \dfrac{HYP,\ \neg P \vdash Q\,,\ HYP,\ \neg P \vdash \neg Q}{HYP \vdash P}\end{array}\right\}\ \text{CONTRA}$$

$$\frac{}{\vdash true}\ \text{TRUE}\qquad \frac{}{\vdash \neg\, false}\ \text{FALSE}$$

A.2 Predicate logic and equality rules

Notes

RT = Referential transparency (of values).

REP = Replacement Theorem, a meta-theorem which asserts the referential transparency of predicates.

Fresh variables introduced by proof rules take arbitrary values.

$P[E/F]$ is predicate P with each occurrence of the expression F replaced by E. $R\langle P/Q\rangle$ is predicate R with some occurrences of the sub-predicate Q replaced by P. Substitutions should not cause variable capture.

A.2.1 Predicate logic rules for FOPLN

$$\frac{HYP,\, \delta(\alpha) \vdash\ P[\alpha/x]}{HYP \vdash\ \forall\ x \bullet P}\ \alpha \text{ fresh} \quad \forall\text{-intro,}$$

$$\frac{HYP \vdash\ \forall\ x \bullet P\,,\ HYP \vdash\ \delta(E)}{HYP \vdash\ P[E/x]} \quad \forall\text{-elim}$$

$$\frac{HYP \vdash\ \delta(E)\,,\ HYP \vdash\ P[E/X]}{HYP \vdash\ \exists\ x \bullet P} \quad \exists\text{-intro}$$

$$\frac{HYP,\, \delta(\alpha) \vdash\ P[\alpha/x] \Rightarrow\ Q,\ HYP,\, \delta(\alpha) \vdash\ \exists\ x \bullet P}{HYP \vdash\ Q}\ \alpha \text{ fresh } x \setminus Q \quad \exists\text{-elim}$$

$$\frac{HYP,\, \delta(\alpha) \vdash\ P[\alpha/X]}{HYP \vdash\ \exists\ x \bullet P}\ \alpha \text{ fresh} \quad \exists\text{-intro-mu (derived law)}$$

A.2.2 Predicate logic rules for FOPL.

$$\frac{HYP \vdash\ P[\alpha/x]}{HYP \vdash\ \forall\ x \bullet P}\ \alpha \text{ fresh } \forall\text{-intro,}$$

$$\frac{HYP \vdash\ \forall\ x \bullet P}{HYP \vdash\ P[E/x]} \quad \forall\text{-elim}$$

$$\frac{HYP \vdash\ P[E/X]}{HYP \vdash\ \exists\ x \bullet P} \quad \exists\text{-intro}$$

$$\frac{HYP \vdash\ P[\alpha/x] \Rightarrow\ Q,\ HYP \vdash\ \exists\ x \bullet P}{HYP \vdash\ Q}\ \alpha \text{ fresh, } x \setminus Q \quad \exists\text{-elim}$$

A.2.3 Equality rules

$$\frac{}{\vdash E = E} \text{ EQUALS}$$

$$\frac{HYP \vdash E = F,\ HYP \vdash P}{HYP \vdash P[E/F]} \text{ RT or LEIBNIZ}$$

$$\frac{HYP \vdash P \Leftrightarrow Q,\ HYP \vdash R}{HYP \vdash R\langle P/Q\rangle} \text{ REP}$$